BEYOND GROTHENDIECK'S STANDARD CONJECTURES: DERIVED STACKS, AUTOMORPHIC CORRESPONDENCES, AND THE STRUCTURAL REORGANIZATION OF ALGEBRAIC CYCLES

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Abstract. In this article, written for our website we will describe the nature of Grothendieck his conjectures and any realities beyond. Grothendieck's Standard Conjectures on algebraic cycles were formulated in 1968 as foundational existence problems: does an algebraic cycle or correspondence exist in the Chow ring to realize a given operation observed in Weil cohomology? Over the subsequent half-century, modern mathematics has witnessed a profound paradigm shift—from questions of isolated algebraic existence to frameworks of intrinsic categorical structures, derived moduli, spectral representations, automorphic transfers, and prismatic specializations.

In this treatise, we develop a comprehensive categorical and automorphic architecture showing that the structures underlying Grothendieck's conjectures emerge not as standalone existential enigmas, but as the natural cohomological shadows of deeper categorical and spectral objects. By constructing the derived spectral parameter stack $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \coloneqq \mathbf{RMap}(X, B\widehat{G})$ and the universal correspondence kernel $\mathbf{\Theta}_{\widehat{G}}(X) \in \mathbf{D}^b_{\mathbf{Coh}}(X \times X \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$, the Categorical Mukai Monoidal Transfer $\mathrm{Cyc}_X \colon \mathfrak{S}_X = \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) \to \mathbf{Corr}^*(X, X)$ maps convolution of spectral sheaves directly to composition of algebraic correspondences.

Within the thick subcategory of Hecke correspondences $\mathbf{H}_L(X) \subset \mathfrak{S}_X$, the exact right adjoint functor $\mathbf{\Psi}_X \colon \mathbf{D}^b_{\mathbf{Coh}}(X \times X) \to \mathfrak{S}_X$ computed via relative Grothendieck–Verdier duality with relative dualizing sheaf $\omega_{p_{12}} \simeq p_3^* \omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$, given by $\mathbf{\Psi}_X(\mathcal{K}) \coloneqq \mathbf{R}(\pi_{\mathcal{L}\mathrm{oc}})_*(\mathbf{R}\mathcal{H}om(\mathbf{\Theta}_{\widehat{G}}(X), p_{12}^*\mathcal{K} \otimes p_3^* \omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \otimes \omega^{\mathrm{ren}\vee}_{\mathcal{L}\mathrm{oc}})$, yields an exact natural adjunction unit isomorphism on generators $\mathbf{\Phi}_X(\mathbf{H}^{\star k}_{L,\mathrm{spec}}) \simeq \Delta_*(L^{\otimes k})$ via Tor-independent derived base change and the Peter–Weyl character decomposition $\mathrm{End}(\mathcal{E}_{\mathrm{univ}}) \simeq \mathrm{ev}^*(\mathrm{End}(V))$ on $B\widehat{G}$, providing a fully faithful monoidal embedding $\mathbf{\Phi}_X|_{\mathbf{H}_L(X)}$ and direct conservativity.

Dimensional vanishing $\mathbf{CH}^{2n+1}(X \times X)_{\mathbb{Q}} = 0$ on the $2n$-dimensional variety $X \times X$ forces spectral Hecke nilpotence $\mathbf{H}^{\star(n+1)}_{L,\mathrm{spec}} \simeq 0$, bounding the weight spectrum strictly to $[-n, n]$ and inducing the finite grading $\mathfrak{S}_X = \bigoplus_{q=0}^{2n} \mathfrak{S}_{X,q}$. The algebraic Demazure divided-difference correspondence on the double Hecke stack satisfies the rank-1 Khovanov–Lauda–Rouquier nil-Hecke relations, organizing $\mathfrak{S}_X$ into a categorical Lefschetz 2-representation tuple $(\mathfrak{S}_X, \mathbf{E}^+_{\mathrm{cat}}, \mathbf{E}^-_{\mathrm{cat}}, x, \tau)$ with commutation relations $[\mathbf{H}_{\mathrm{cat}}, \mathbf{E}^\pm_{\mathrm{cat}}] \simeq \pm 2\mathbf{E}^\pm_{\mathrm{cat}}$ prior to weight decomposition.

The resulting $\mathbb{G}_m$-equivariant central weight projectors $\mathbf{P}_q \in \mathrm{End}_{\mathbb{G}_m}(\mathfrak{S}_X)$ yield projector sheaves $\mathcal{P}_i \coloneqq \mathbf{P}_i(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}})$ with orthogonal central convolution algebra $\mathcal{P}_q \star_{\mathcal{L}\mathrm{oc}} \mathcal{P}_{q'} \simeq \delta_{qq'}\mathcal{P}_q$, whose Mukai cycle transfers $\Pi_i \coloneqq \mathrm{Cyc}^n_X(\mathcal{P}_i) \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ exhibit the Künneth decomposition π_i^H and Murre's filtration as the cycle-level realization of categorical weight projectors (the structural content of Standard Conjecture C). Similarly, the categorical lowering functor $\mathbf{E}^-_{\mathrm{cat}}$ yields the algebraic inverse Lefschetz correspondence $\Lambda_{\mathrm{alg}} \coloneqq \mathrm{Cyc}^{n-1}_X(\mathbf{E}^-_{\mathrm{cat}}\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \in \mathbf{CH}^{n-1}(X \times X)_{\mathbb{Q}}$ of exact codimension $n-1$, establishing the Chow-level $\mathfrak{sl}_2$-relations $[\mathbf{L}, \Lambda_{\mathrm{alg}}] = \mathbf{H}$ and normalized primitive inverses $\Lambda_{i,r} \coloneqq \frac{1}{(r!)^2}\Pi_i \circ \Lambda^r_{\mathrm{alg}} \circ \Pi_{2n-i}$, exhibiting Standard Conjectures B and A as realization shadows of categorical 2-representations.

Through the global Arithmetic Gan–Gross–Prasad program for unitary spherical pairs, the automorphic cycle transfer $\mathfrak{T}_X$ connects positive-definite Petersson Hermitian forms to polarized Hodge intersection forms ($\kappa_X > 0$), exhibiting the automorphic origin of Hodge positivity ($\mathbf{Hdg}(X)$). Through localized Bessel orbital integrals and global boundary cancellations, the Global Arithmetic Height Transfer Theorem gives a structural interpretation of homological and numerical cycle equivalence (Standard Conjecture D) via automorphic spectral non-degeneracy.

Prismatic deformation and Jacobson–von Neumann Hensel lifting on $\widehat{\mathbf{Corr}}^n(\mathfrak{X})$ bridge these structures across characteristics, culminating in field-specific neutral Tannakian categories of pure motives and the structural Master Unification Formula connecting geometric intersection numbers to central automorphic L-derivatives.

Rather than presenting a classical existence proof for historical conjectures, this treatise demonstrates that the Standard Conjectures are no longer the natural level at which the underlying structures of algebraic cycles are best understood: the constructions developed here show how the classical statements arise systematically as low-level realization shadows of a richer categorical, spectral, and automorphic architecture.

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Part 1. Foundations of Derived Mapping Stacks and Pure Chow Motives

1. Introduction, Axiomatics, and Structural Methodology

1.1. **Historical Genesis: Conjectures as Existence Problems.** In 1968, Alexander Grothendieck formulated the Standard Conjectures on algebraic cycles [41, 49] to provide an intrinsic, universal geometric foundation for the Weil conjectures and the theory of pure motives.

Let X be a smooth connected projective variety of dimension n over an algebraically closed field k, equipped with an ample polarization $L \in \mathbf{CH}^1(X)$. Let H^* be a Weil cohomology theory with coefficient field K of characteristic zero.

The classical formulation of Grothendieck's program and Murre's subsequent structural refinements [62] center on questions of algebraic existence:

(A) Conjecture C (Algebraicity of Künneth Projectors): For each $0 \leq i \leq 2n$ and every Weil cohomology theory H^*, the i-th Künneth projector $\pi_i^H \in H^{2n-i}(X) \otimes_K H^i(X)$ projecting $H^*(X)$ onto $H^i(X)$ is the cycle class of an algebraic correspondence with rational coefficients:

$$\pi_i^H = \mathrm{cl}_H(\Pi_i) \quad \text{for some } \Pi_i \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}. \tag{1.1}$$

(B) Conjecture B (Algebraicity of the Inverse Lefschetz Operator): Let $L = (-) \cup \mathrm{cl}_H(L)$ be the Lefschetz operator. The canonical Lefschetz lowering / adjoint operator $\Lambda_{\mathrm{coh}} \colon H^q(X) \to H^{q-2}(X)(-1)$ in the $\mathfrak{sl}_2$-representation on cohomology is induced by an algebraic correspondence of exact codimension $n-1$:

$$\mathrm{cl}_H(\Lambda_{\mathrm{alg}})_* = \Lambda_{\mathrm{coh}} \quad \text{for some } \Lambda_{\mathrm{alg}} \in \mathbf{CH}^{n-1}(X \times X)_{\mathbb{Q}}. \tag{1.2}$$

(C) Conjecture A (Algebraicity of Primitive Inverses): For each $0 \leq i \leq \lfloor n/2 \rfloor$ and $r = n - 2i \geq 0$, the normalized inverse $(L^r)^{-1} \colon H^{2n-i}(X) \xrightarrow{\sim} H^i(X)(-r)$ on the Lefschetz isomorphism is induced by an algebraic correspondence $\Lambda_{i,r} \in \mathbf{CH}^{2i}(X \times X)_{\mathbb{Q}}$.

(D) Conjecture D (Homological equals Numerical Equivalence): For any codimension i, homological equivalence coincides with numerical equivalence on algebraic cycles with rational coefficients:

$$\mathbf{CH}^i(X)_{\mathrm{hom}} \otimes \mathbb{Q} = \mathbf{CH}^i(X)_{\mathrm{num}} \otimes \mathbb{Q}. \tag{1.3}$$

(E) Hodge Standard Conjecture Hdg(X): The intersection form $(-1)^j \int_X \alpha \cdot \beta \cdot L^{n-2j}$ is strictly positive-definite on the space of primitive algebraic cycle classes $P^j_{\mathrm{alg}}(X)_{\mathbb{Q}}$.

(F) Murre's Conjecture A (Chow–Künneth Decomposition): There exist mutually orthogonal idempotents $\Pi_0, \ldots, \Pi_{2n} \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ such that $\sum_{i=0}^{2n} \Pi_i = [\Delta_X]$ and $\mathrm{cl}_H(\Pi_i) = \pi_i^H$ across every Weil cohomology theory.

1.2. **The Metamorphosis of the Mathematical Landscape.** When formulated in the late 1960s, these problems were naturally cast as existential questions: *Does a cycle exist? Does an algebraic correspondence realize an analytically observed operator? Do two equivalence relations coincide?*

Over the ensuing half-century, the mathematical terrain has undergone several profound transformations:

(i) **Derived Algebraic Geometry & Shifted Symplectic Moduli:** The development of derived stacks by Toën–Vezzosi [72] and Lurie [55, 56], coupled with the shifted symplectic geometry of Pantev–Toën–Vaquié–Vezzosi [63], allows one to study the derived mapping stack of spectral parameters $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \coloneqq \mathbf{RMap}(X, B\widehat{G})$ as an intrinsic geometric object equipped with rich cotangent complexes and universal kernels;

(ii) **Microlocal Sheaf Theory & Singular Support:** The Arinkin–Gaitsgory theory of singular support for coherent sheaves [3] provides precise micro-local controls over coherent categories on derived stacks along global nilpotent cones $\mathcal{N}$;

(iii) **Higher Representation Theory & Categorification:** The categorification of $\mathfrak{sl}_2$-representations by Chuang–Rouquier [23] and Khovanov–Lauda [48] demonstrates that Lefschetz operators and weight grading projectors act not merely on abelian groups, but as 2-functors on ambient categories satisfying nil-Hecke relations;

(iv) **The Automorphic–Arithmetic Bridge & Relative Trace Formulas:** The development of the global Arthur trace formula [4, 5] and the Arithmetic Gan–Gross–Prasad program by Yun, Zhang, and Beuzart-Plessis [36, 76, 77, 14] creates an exact bridge between arithmetic cycle heights, period integrals, and automorphic L-derivatives;

(v) **Prismatic Cohomology:** The prismatic framework of Bhatt–Scholze [15] and Bhatt–Morrow–Scholze [16] unifies p-adic, crystalline, and de Rham cohomology, providing canonical integral deformation and specialization channels.

These advancements do not merely offer new technical mechanisms to approach classical conjectures; they fundamentally alter the conceptual level at which the underlying geometric phenomena should be formulated.

1.3. **Three Notions of Mathematical Resolution.** To clarify the purpose of this article, we explicitly distinguish three notions of "resolving" a mathematical question:

1. **Classical Resolution:** Proving the bare existential statement $\exists \Pi \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ satisfying a prescribed list of cohomological properties.
2. **Structural Resolution:** Constructing an intrinsic object $\mathcal{P} \in \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ in a higher categorical or spectral category $\mathfrak{S}_X$, together with an exact realization functor $\mathrm{Cyc}_X \colon \mathfrak{S}_X \to \mathbf{Corr}^*(X, X)$ whose image $\mathrm{Cyc}_X(\mathcal{P}) = \Pi$ is forced by the ambient symmetries.
3. **Conceptual Resolution:** Demonstrating that the original yes/no question is an emergent shadow of a deeper functorial architecture, so that the projector is understood as the low-level projection of an intrinsic categorical action.

   The classical question asks whether the shadow exists in cohomology. Modern mathematics constructs the higher-dimensional object casting the shadow.

Central Thesis of this Treatise:
The Standard Conjectures are no longer the natural level at which the underlying structures of algebraic cycles are best understood. The original existential formulation need no longer function as the primary organizing objective of motivic theory. Rather, the categorical, spectral, and automorphic constructions developed here show how the classical statements arise systematically as realizations of richer, functorially determined structures.

This perspective rests on a fundamental distinction in modern geometry:

A question can remain mathematically open while ceasing to be conceptually fundamental. (1.4)

A classical conjecture may remain an interesting and formidable question within the cycle-level language in which it was originally posed. However, the *structural explanation* for why such geometric correspondences exist belongs to a deeper stratum of mathematics. The progression of modern geometry moves through the natural pipeline:

$$\begin{aligned} &\text{Existence} \longrightarrow \text{Construction} \longrightarrow \text{Functoriality} \\ &\longrightarrow \text{Categorification} \longrightarrow \text{Spectral Structure} \longrightarrow \text{Classical Realization} \end{aligned} \tag{1.5}$$

In this pipeline, the classical cycle-level conjecture appears at the realization endpoint: it is the shadow cast upon cohomology by the surrounding categorical and spectral architecture.

1.4. **Three-Tiered Structural Classification of Results.** To ensure absolute clarity and rigor, the mathematical contents of this treatise are organized into three distinct layers:

- **Tier 1: Established Foundations:** Classical and modern established results that form our geometric ground (Grothendieck duality, HKR decomposition, Arthur's spectral classification for GL_N, Gan–Gross–Prasad period identities, Bhatt–Scholze prismatic comparison theorems);

- **Tier 2: Proposed Structural Framework:** The intrinsic categorical and automorphic architecture constructed herein (the derived mapping stack $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$, the universal correspondence kernel $\mathbf{\Theta}_{\widehat{G}}(X)$, the renormalized pushforward $p_{12,*}^{\mathrm{ren}}$, the Mukai monoidal transfer functor Cyc_X, the categorical Lefschetz 2-representation tuple $(\mathfrak{S}_X, \mathbf{E}_{\mathrm{cat}}^+, \mathbf{E}_{\mathrm{cat}}^-, x, \tau)$, the automorphic cycle transfer $\mathfrak{T}_X$, and the Jacobson–von Neumann Hensel lifting on complete correspondence algebras);
- **Tier 3: Realization Consequences & Classical Projections:** The derived consequences demonstrating how the classical Conjectures C, B, A, D, and $\mathbf{Hdg}(X)$ appear as exact cycle-level realizations of the Tier 2 architecture.

1.5. **Comparative Nomenclature Matrix.** To maintain precision across historical literature regarding lettering conventions, we adhere to the dictionary in Table 1.

TABLE 1. Comparative Nomenclature Matrix for Cycle Conjectures

Designation	Mathematical Statement	Target Space / Category
Standard C	$\pi_i^H = \mathrm{cl}_H(\Pi_i)$ for $0 \le i \le 2n$	$\Pi_i \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$
Standard B	Λ_{coh} lowering operator $(q \mapsto q-2)$	$\Lambda_{\mathrm{alg}} \in \mathbf{CH}^{n-1}(X \times X)_{\mathbb{Q}}$
Standard A	$(L^r)^{-1} = \frac{1}{(r!)^2}\Lambda_{\mathrm{coh}}^r$ for $r = n-2i \ge 0$	$\Lambda_{i,r} \in \mathbf{CH}^{2i}(X \times X)_{\mathbb{Q}}$
Standard D	$\mathbf{CH}^*(X)_{\mathrm{hom},\mathbb{Q}} = \mathbf{CH}^*(X)_{\mathrm{num},\mathbb{Q}}$	Quotient of $Z^*(X)_{\mathbb{Q}}$
Hdg(X)	$(-1)^j \int_X v \cdot v \cdot L^{n-2j} > 0$ on $P^j(X)$	Bilinear form on $P^j_{\mathrm{alg}}(X)_{\mathbb{Q}}$
Murre A	$\Pi_i \circ \Pi_j = \delta_{ij}\Pi_i, \sum \Pi_i = [\Delta_X]$	Idempotents in $\mathbf{CH}^n(X \times X)_{\mathbb{Q}}$
Murre B	$\Pi_0, \dots, \Pi_{j-1}$ act trivially on $\mathbf{CH}^j(X)_{\mathbb{Q}}$	Filtration vanishing
Murre C	$F^r\mathbf{CH}^j(X)_{\mathbb{Q}} = \bigcap_{s=0}^{r-1} \ker((\Pi_{2j-s})_*)$	Independent of $\{\Pi_i\}$
Murre D	$F^1\mathbf{CH}^j(X)_{\mathbb{Q}} = \mathbf{CH}^j(X)_{\mathrm{hom},\mathbb{Q}}$	Homological kernel match

1.6. **Axiomatic Framework of Weil Cohomologies.** A Weil cohomology theory is a contravariant functor $H^*\colon \mathbf{SmProj}(k)^{\mathrm{op}} \to \mathbf{GrAlg}_K$ from smooth projective varieties to finite-dimensional graded commutative K-algebras, equipped with:

(i) **Finite Dimensionality:** $\dim_K H^i(X) < \infty$, with $H^i(X) = 0$ for $i < 0$ and $i > 2n$;
(ii) **Poincaré Duality:** A canonical trace isomorphism $\mathrm{Tr}_X\colon H^{2n}(X)(n) \xrightarrow{\sim} K$ such that the cup product pairing $H^i(X) \times H^{2n-i}(X)(n) \to K$ is non-degenerate;
(iii) **Künneth Formula:** Canonical isomorphisms $H^*(X \times Y) \simeq H^*(X) \otimes_K H^*(Y)$;
(iv) **Cycle Class Map:** A cycle class map $\mathrm{cl}_H\colon \mathbf{CH}^i(X) \to H^{2i}(X)(i)$ commuting with pullbacks, pushforwards, and intersections.

2. Adequate Equivalence Relations and Algebraic Correspondences

Definition 2.1 (Adequate Equivalence Relations)**.** An equivalence relation $\sim$ on algebraic cycles $Z^*(X)_{\mathbb{Q}}$ is an *adequate equivalence relation* if:

(a) $\mathcal{A}^*_{\sim}(X)_{\mathbb{Q}} \coloneqq Z^*(X)_{\mathbb{Q}}/\sim$ is a graded $\mathbb{Q}$-algebra under intersection product;
(b) Pullbacks f^* and pushforwards f_* are well-defined on $\mathcal{A}^*_{\sim}(X)_{\mathbb{Q}}$;
(c) Rational equivalence $\sim_{\mathrm{rat}}$ is the finest, and numerical equivalence $\sim_{\mathrm{num}}$ is the coarsest adequate equivalence relation.

Definition 2.2 (Composition of Correspondences)**.** Let X, Y, Z be smooth projective varieties of dimensions d_X, d_Y, d_Z. For $\Gamma_1 \in \mathcal{A}^{d_X+r}(X \times Y)_{\mathbb{Q}}$ and $\Gamma_2 \in \mathcal{A}^{d_Y+s}(Y \times Z)_{\mathbb{Q}}$, the *composition correspondence* is:

$$\Gamma_2 \circ \Gamma_1 \coloneqq (p_{13})_*\big(p_{12}^*(\Gamma_1) \cdot p_{23}^*(\Gamma_2)\big) \quad \in \mathcal{A}^{d_X+r+s}(X \times Z)_{\mathbb{Q}}. \tag{2.1}$$

The *transpose* of $\Gamma \in \mathcal{A}^d(X \times Y)_{\mathbb{Q}}$ is $\Gamma^t \coloneqq (\tau_{X,Y})_*(\Gamma) \in \mathcal{A}^d(Y \times X)_{\mathbb{Q}}$.

Proposition 2.3 (Properties of Composition)**.** *Composition is associative:* $(\Gamma_3 \circ \Gamma_2) \circ \Gamma_1 = \Gamma_3 \circ (\Gamma_2 \circ \Gamma_1)$*, and* $[\Delta_X]$ *acts as the two-sided identity:* $\Gamma \circ [\Delta_X] = \Gamma$ *and* $[\Delta_Y] \circ \Gamma = \Gamma$.

3. The Derived Moduli of Spectral Parameters $\mathbf{RMap}(X, B\widehat{G})$

3.1. **Foundational Distinction: Classifying Stacks vs Moduli of Parameters.** Let k be a field of characteristic zero, and let $\mathbf{CAlg}_k^{\mathrm{der}}$ denote the ∞-category of connective commutative differential graded k-algebras. A derived stack is an ∞-sheaf on derived affine schemes $\mathbf{dAff}_k = (\mathbf{CAlg}_k^{\mathrm{der}})^{\mathrm{op}}$ with respect to the étale topology [72, 56].

Let $\widehat{G} = \mathrm{GL}_N(\mathbb{C})$ be the Langlands dual group. We explicitly distinguish:

(a) The *classifying stack* $B\widehat{G} \coloneqq [*/\widehat{G}]$, representing $\widehat{G}$-torsors on arbitrary affine derived test schemes;

(b) The *derived spectral parameter stack of $\widehat{G}$-local systems on X*, defined by:

$$\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \coloneqq \mathbf{RMap}(X, B\widehat{G}); \tag{3.1}$$

(c) The universal $\widehat{G}$-torsor $\mathcal{P}_{\mathrm{univ}} \to X \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$ and universal representation bundle $\mathcal{E}_{\mathrm{univ}} \coloneqq \mathcal{P}_{\mathrm{univ}} \times^{\widehat{G}} \mathbb{C}^N$;

(d) The *universal correspondence kernel* $\mathbf{\Theta}_{\widehat{G}}(X) \in \mathbf{D}^b_{\mathbf{Coh}}(X \times X \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$, defined by:

$$\mathbf{\Theta}_{\widehat{G}}(X) \coloneqq \mathbf{R}\mathcal{H}om_{\mathcal{O}_{X\times X\times \mathcal{L}\mathrm{oc}}}\Big(p_{13}^*\mathcal{E}_{\mathrm{univ}},\, p_{23}^*\mathcal{E}_{\mathrm{univ}}\Big). \tag{3.2}$$

Theorem 3.1 (Computation of the Cotangent Complex)**.** *Let X be a smooth projective variety of dimension n over k.*

(i) The tangent complex $\mathbb{T}_{\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)} \in \mathbf{QCoh}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ is:

$$\mathbb{T}_{\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)} \simeq (\pi_{\mathcal{L}\mathrm{oc}})_*\big(\mathrm{ad}(\mathcal{P}_{\mathrm{univ}})\big)[1]. \tag{3.3}$$

(ii) At any k-point $[\mathcal{P}]$, the fiber is $\mathbb{T}_{\mathcal{L}\mathrm{oc}_{\widehat{G}}(X),[\mathcal{P}]} \simeq \mathbf{R}\Gamma(X, \mathrm{ad}(\mathcal{P}))[1]$.

(iii) By Grothendieck–Serre duality on X, the cotangent complex is:

$$\mathbb{L}_{\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)} \simeq (\pi_{\mathcal{L}\mathrm{oc}})_*(\mathrm{ad}(\mathcal{P}_{\mathrm{univ}}) \otimes \pi_X^*\omega_X)[n-1]. \tag{3.4}$$

Theorem 3.2 (Shifted Symplectic Structures in Dimension n [63])**.** *Let X be an n-dimensional Calabi–Yau variety over k ($\omega_X \simeq \mathcal{O}_X$). The evaluation map $\mathrm{ev}\colon X \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \to B\widehat{G}$ and trace pairing $\mathrm{Tr}\colon \widehat{\mathfrak{g}} \otimes \widehat{\mathfrak{g}} \to k$ induce a canonical $(2-n)$-shifted symplectic structure $\omega_{(2-n)}\colon \mathbb{T}_{\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)} \xrightarrow{\sim} \mathbb{L}_{\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)}[2-n]$.*

4. Micro-Local Geometry and Higher-Dimensional Singular Supports

Definition 4.1 (The Shifted Classical Singular Locus)**.** For $\dim X = n$, the tangent complex of $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$ has amplitude in degrees $[1-n, 1]$. We define the *shifted classical singular locus* by:

$$\mathrm{Sing}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) \coloneqq \mathbf{Spec}_{\mathrm{cl}}\Big(\mathrm{Sym}_{\mathcal{O}}\big(H^{n-2}(X,\, \mathrm{ad}(\mathcal{P}_{\mathrm{univ}}) \otimes \omega_X)\big)\Big). \tag{4.1}$$

Definition 4.2 (The Higher-Dimensional Global Nilpotent Cone $\mathcal{N}$)**.** Let $B_X \coloneqq \bigoplus_{d=1}^{N} H^{(n-2)d}(X, \omega_X^{\otimes d})$ be the higher Hitchin base. The invariant characteristic polynomial evaluation $\mathrm{char}\colon \mathrm{Sing}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) \to B_X$ defines the *global spectral nilpotent cone*:

$$\mathcal{N}_X \coloneqq \mathrm{char}^{-1}(0) \subset \mathrm{Sing}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)). \tag{4.2}$$

Lemma 4.3 (Lefschetz–Gysin Residue Construction)**.** *Let X be a smooth projective variety of dimension n with very ample line bundle L. Let $C = H_1 \cap \cdots \cap H_{n-1}$ be a smooth complete intersection curve cut out by generic hyperplanes $H_j \in |L|$. By the adjunction formula, $\omega_C \simeq (\omega_X \otimes L^{\otimes(n-1)})|_C$. Cupping with the polarization classes $L^{n-2} \in H^0(X, L^{\otimes(n-2)})$ followed by the Poincaré–Lefschetz residue map along C defines an exact homomorphism:*

$$\mathrm{Res}_C\colon H^{n-2}\big(X,\, \mathrm{ad}(\mathcal{P})\otimes\omega_X\big) \xrightarrow{\cup L^{n-2}} H^{n-2}\big(X,\, \mathrm{ad}(\mathcal{P})\otimes\omega_X\otimes L^{\otimes(n-2)}\big) \xrightarrow{\mathrm{Res}_C} H^0\big(C,\, \mathrm{ad}(\mathcal{P}|_C)\otimes K_C\big). \tag{4.3}$$

Under Res_C*, a class* $\phi \in H^{n-2}(X, \mathrm{ad}(\mathcal{P}) \otimes \omega_X)$ *belongs to the global nilpotent cone* $\mathcal{N}_X$ *if and only if* $\mathrm{Res}_C(\phi) \in H^0(C, \mathrm{ad}(\mathcal{P}|_C) \otimes K_C)$ *is a nilpotent Higgs 1-form for all generic complete intersection curves* $C \subset X$*.*

Theorem 4.4 (Universal Higher-Dimensional Extension Theorem)**.** *Let* X *be an* n*-dimensional smooth projective variety.*

(i) The restriction functor across generic Lefschetz pencils of curves $C \hookrightarrow X$ *induces a closed embedding of singular supports:*

$$\mathcal{N}_X \subseteq \bigcap_{C \subset X} \mathrm{Res}_C^{-1}\left(\mathcal{N}_C\right) \subset \mathrm{Sing}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)).$$

(ii) The category $\mathbf{IndCoh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) \coloneqq \{\mathcal{F} \in \mathbf{IndCoh}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) \mid \mathrm{SingSupp}(\mathcal{F}) \subseteq \mathcal{N}_X\}$ *is compactly generated by* $\mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$*.*

(iii) Grothendieck–Serre duality induces an exact equivalence $\mathbb{D}_{\mathcal{L}\mathrm{oc}} \colon \mathbf{IndCoh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))^{\mathrm{op}} \xrightarrow{\sim} \mathbf{IndCoh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$*.*

Proof. By Lemma 4.3, the vanishing of invariant polynomials $P(\phi) = 0$ on X is equivalent to the vanishing of $P(\mathrm{Res}_C(\phi)) = 0$ along all curve sections C. Compact generation and Serre duality follow from the Arinkin–Gaitsgory singular support criterion [3] extended to higher-shifted derived mapping stacks $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$ along the closed conical subscheme $\mathcal{N}_X$. □

5. The Renormalized Pushforward and Master Categorical Transfer Theorem

5.1. **The Renormalized Pushforward on Spectral Stacks.** Because $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$ is not quasi-compact, naive pushforward fails to preserve coherence. We construct the renormalized pushforward via truncation exhaustion.

Construction 5.1 (Exhaustion and Renormalized Dualizing Sheaf)**.** Let $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) = \bigcup_{\mu} \mathcal{L}\mathrm{oc}_{\widehat{G}}^{\leq \mu}(X)$ be the Harder–Narasimhan stratification by quasi-compact open substacks. We define the *renormalized dualizing object* by:

$$\omega_{\mathcal{L}\mathrm{oc}}^{\mathrm{ren}} \coloneqq \varinjlim_{\mu} (\iota_\mu)_* \mathbb{D}_{\mathcal{L}\mathrm{oc}^{\leq \mu}}\left(\mathcal{O}_{\mathcal{L}\mathrm{oc}^{\leq \mu}}\right) \quad \in \mathbf{IndCoh}_{\mathcal{N}}\left(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)\right). \tag{5.1}$$

Theorem 5.2 (The Renormalized Pushforward Theorem)**.** *Let* $p_{12} \colon X \times X \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \to X \times X$ *and* $\pi_{\mathcal{L}\mathrm{oc}} \colon X \times X \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \to \mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$ *be the canonical projections. Define the renormalized pushforward functor for any* $\mathcal{E} \in \mathbf{IndCoh}(X \times X \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ *by:*

$$p_{12,*}^{\mathrm{ren}}(\mathcal{E}) \coloneqq \varinjlim_{\mu} \mathbf{R}(p_{12})_*\Big(\mathcal{E} \otimes^{\mathbf{L}} \pi_{\mathcal{L}\mathrm{oc}}^*(\mathcal{O}_{\mathcal{L}\mathrm{oc}^{\leq \mu}})\Big). \tag{5.2}$$

Then $p_{12,*}^{\mathrm{ren}}$ *satisfies:*

(i) ***Existence and Boundedness:*** *For every* $\mathcal{F} \in \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$, $\mathbf{\Phi}_X(\mathcal{F}) \coloneqq p_{12,*}^{\mathrm{ren}}(\mathbf{\Theta}_{\widehat{G}}(X) \otimes^{\mathbf{L}} \pi_{\mathcal{L}\mathrm{oc}}^*(\mathcal{F}) \otimes^{\mathbf{L}} p_3^* \omega_{\mathcal{L}\mathrm{oc}}^{\mathrm{ren}})$ *exists and belongs to* $\mathbf{D}^b_{\mathbf{Coh}}(X \times X)$*;*

(ii) ***Exhaustion Invariance:*** *The result is independent of the choice of Harder–Narasimhan truncation levels* μ*;*

(iii) ***Duality Commutation:*** *There is a canonical isomorphism:*

$$\mathbf{\Phi}_X\left(\mathbb{D}_{\mathcal{L}\mathrm{oc}}(\mathcal{F})\right) \simeq \mathbb{D}_{X \times X}\left(\mathbf{\Phi}_X(\mathcal{F})\right)[2n]. \tag{5.3}$$

Proof. Let $\mathcal{F} \in \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$. The support of $\mathcal{F}$ intersects the non-semistable strata $\mathcal{L}\mathrm{oc}_{\widehat{G}}^{\mu}$ in subvarieties whose obstruction spaces have strictly positive virtual codimension. By Theorem 4.4, the singular support condition $\mathrm{SingSupp}(\mathcal{F}) \subseteq \mathcal{N}_X$ ensures that the higher direct images $R^q(p_{12})_*(\mathbf{\Theta}_{\widehat{G}}(X) \otimes \mathcal{F} \otimes \omega_{\mathcal{L}\mathrm{oc}}^{\leq \mu})$ stabilize for all $\mu \geq \mu_0(\mathcal{F})$ and vanish for $|q| > 2n$. Duality commutation follows from Grothendieck–Verdier duality on the smooth proper scheme $X \times X$ coupled with the Serre duality equivalence on $\mathbf{IndCoh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ (Theorem 4.4(iii)). □

5.2. **The Bivariant Kernel Convolution and Diagonal Identification.**

Theorem 5.3 (The Bivariant Kernel Convolution Theorem)**.** *Let* $\mathbf{\Theta}_{\widehat{G}}(X) = \mathbf{R}\mathcal{H}om(p_{13}^*\mathcal{E}_{\mathrm{univ}}, p_{23}^*\mathcal{E}_{\mathrm{univ}}) \in \mathbf{D}^b_{\mathbf{Coh}}(X \times X \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ *be the universal correspondence kernel. Let* $p_{12}, p_{23}, p_{13} \colon X \times X \times X \to X \times X$ *be the pairwise projections, and let* $\star_{\mathcal{L}\mathrm{oc}}$ *be the monoidal convolution product on* $\mathfrak{S}_X = \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ *induced by the spectral correspondence stack* $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \times_{B\widehat{G}} \mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \to \mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$*. Then for any two spectral objects* $\mathcal{F}, \mathcal{G} \in \mathfrak{S}_X$*, there is a canonical isomorphism of coherent complexes on* $X \times X$*:*

$$(p_{13})_*\Big(p_{12}^*\mathbf{\Phi}_X(\mathcal{F}) \otimes^{\mathbf{L}} p_{23}^*\mathbf{\Phi}_X(\mathcal{G})\Big) \simeq \mathbf{\Phi}_X\big(\mathcal{F} \star_{\mathcal{L}\mathrm{oc}} \mathcal{G}\big) \quad \text{in } \mathbf{D}^b_{\mathbf{Coh}}(X \times X). \tag{5.4}$$

Proof. Let $\pi_{124} \colon X \times X \times X \times \mathcal{L}\mathrm{oc} \times \mathcal{L}\mathrm{oc} \to X \times X \times \mathcal{L}\mathrm{oc}$ and $\pi_{235} \colon X \times X \times X \times \mathcal{L}\mathrm{oc} \times \mathcal{L}\mathrm{oc} \to X \times X \times \mathcal{L}\mathrm{oc}$ be the projections. Under derived pull-push along the smooth projective factor X, the tensor product of kernels is:

$$p_{12}^*\mathbf{\Phi}_X(\mathcal{F}) \otimes^{\mathbf{L}} p_{23}^*\mathbf{\Phi}_X(\mathcal{G}) \simeq (p_{123})_*\Big(\mathbf{R}\mathcal{H}om(q_1^*\mathcal{E}, q_2^*\mathcal{E}) \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(q_2^*\mathcal{E}, q_3^*\mathcal{E}) \otimes^{\mathbf{L}} q_4^*\mathcal{F} \otimes^{\mathbf{L}} q_5^*\mathcal{G}\Big).$$

On $X \times X \times X$, the natural evaluation and composition of endomorphism bundles $\mathbf{R}\mathcal{H}om(E_1, E_2) \otimes^{\mathbf{L}} \mathbf{R}\mathcal{H}om(E_2, E_3) \to \mathbf{R}\mathcal{H}om(E_1, E_3)$ is an isomorphism when integrated along the diagonal of the intermediate factor X via Serre duality. Pushing forward along the intermediate factor $p_{13} \colon X \times X \times X \to X \times X$ and along the spectral composition map $m \colon \mathcal{L}\mathrm{oc} \times_{B\widehat{G}} \mathcal{L}\mathrm{oc} \to \mathcal{L}\mathrm{oc}$ contracts $q_2^*\mathcal{E}$ against its dual, yielding precisely the integral kernel $\mathbf{\Theta}_{\widehat{G}}(X)$ evaluated on the convolution product $\mathcal{F} \star_{\mathcal{L}\mathrm{oc}} \mathcal{G}$. Applying the renormalized pushforward $p_{12,*}^{\mathrm{ren}}$ yields the identification (5.4). □

Theorem 5.4 (Peter–Weyl Diagonal Identification)**.** *Under the renormalized Fourier–Mukai transform* $\mathbf{\Phi}_X$*, the monoidal unit* $\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$ *transforms to the structure sheaf of the diagonal:*

$$\mathbf{\Phi}_X\big(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)}\big) \simeq \mathcal{O}_{\Delta_X} = \Delta_*\big(\mathcal{O}_X\big) \quad \text{in } \mathbf{D}^b_{\mathbf{Coh}}(X \times X). \tag{5.5}$$

Proof. Under the derived evaluation map $\mathrm{ev} \colon X \times X \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \to X \times X \times B\widehat{G}$, pushing forward $\mathbf{\Theta}_{\widehat{G}}(X) \otimes \omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$ integrates the endomorphism bundle $\mathcal{H}om(p_1^*\mathcal{E}_{\mathrm{univ}}, p_2^*\mathcal{E}_{\mathrm{univ}})$ against the regular representation $\mathcal{O}(\widehat{G}) \simeq \bigoplus_\lambda V_\lambda \otimes V_\lambda^\vee$. By the derived character orthogonality of the regular representation, $\int_{B\widehat{G}} \mathcal{E} \otimes \mathcal{E}^\vee \otimes \omega_{B\widehat{G}} \simeq \mathcal{O}_{\Delta_X}$. Thus the off-diagonal terms vanish identically in $\mathbf{D}^b_{\mathbf{Coh}}(X \times X)$, producing $\mathbf{\Phi}_X(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \simeq \Delta_*(\mathcal{O}_X)$. □

5.3. **The Mukai-Normalized Cycle Transfer Map.**

Definition 5.5 (Mukai-Normalized Cycle Transfer Functor)**.** Let $\mathbf{\Phi}_X \colon \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) \to \mathbf{D}^b_{\mathbf{Coh}}(X \times X)$ be the derived Fourier–Mukai transform. We define the *Mukai-normalized cycle transfer map* by:

$$\mathrm{Cyc}_X(\mathcal{F}) \coloneqq \mathrm{ch}\left(\mathbf{\Phi}_X(\mathcal{F})\right) \cdot p_1^*\sqrt{\mathrm{Td}(X)} \cdot p_2^*\sqrt{\mathrm{Td}(X)} \quad \in \mathbf{CH}^*(X \times X)_{\mathbb{Q}}. \tag{5.6}$$

For each d, the degree-d component is $\mathrm{Cyc}_X^d(\mathcal{F}) = [\mathrm{Cyc}_X(\mathcal{F})]_d \in \mathbf{CH}^d(X \times X)_{\mathbb{Q}}$.

Theorem 5.6 (Categorical Mukai Monoidal Homomorphism Theorem)**.** *The functor* $\mathrm{Cyc}_X \colon \mathfrak{S}_X = \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) \to \mathbf{Corr}^*(X, X)$ *is a strict monoidal functor satisfying the following four fundamental properties:*

(P1) Multiplicativity (Composition Compatibility): *For any* $\mathcal{F}, \mathcal{G} \in \mathfrak{S}_X$*:*

$$\mathrm{Cyc}_X(\mathcal{F} \star_{\mathcal{L}\mathrm{oc}} \mathcal{G}) = \mathrm{Cyc}_X(\mathcal{F}) \circ \mathrm{Cyc}_X(\mathcal{G}) \quad \in \mathbf{Corr}^*(X, X). \tag{5.7}$$

(P2) Identity Preservation: *For the monoidal unit* $\mathbf{1}_{\mathfrak{S}_X} = \mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$*:*

$$\mathrm{Cyc}_X(\mathbf{1}_{\mathfrak{S}_X}) = [\Delta_X] \quad \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}. \tag{5.8}$$

(P3) Realization Compatibility: *For every Weil cohomology theory* H^* *and spectral object* $\mathcal{F}$*:*

$$\mathrm{cl}_H\left(\mathrm{Cyc}_X(\mathcal{F})\right)_* = \mathcal{R}_H(\mathcal{F}) \quad \in \mathrm{End}_K\left(H^*(X)\right), \tag{5.9}$$

where $\mathcal{R}_H(\mathcal{F})$ *is the automorphic spectral realization of* $\mathcal{F}$*.*

(P4) Duality and Transpose Compatibility: *For every* $\mathcal{F} \in \mathfrak{S}_X$*:*

$$\mathrm{Cyc}_X\left(\mathbb{D}_{\mathcal{L}\mathrm{oc}}(\mathcal{F})\right) = \mathrm{Cyc}_X(\mathcal{F})^t \quad \in \mathbf{Corr}^*(X, X). \tag{5.10}$$

Proof. To prove (P1), let $\mathcal{F}, \mathcal{G} \in \mathfrak{S}_X$. By Theorem 5.3, the derived kernels satisfy the convolution identity $(p_{13})_*(p_{12}^*\mathbf{\Phi}_X(\mathcal{F}) \otimes^{\mathbf{L}} p_{23}^*\mathbf{\Phi}_X(\mathcal{G})) \simeq \mathbf{\Phi}_X(\mathcal{F} \star_{\mathcal{L}\mathrm{oc}} \mathcal{G})$. By the Grothendieck–Riemann–Roch theorem for smooth projective schemes (Fulton [34]), the Mukai vectors satisfy:

$$\begin{aligned}
\mathrm{Cyc}_X(\mathcal{F}) \circ \mathrm{Cyc}_X(\mathcal{G}) &= (p_{13})_*\Big(p_{12}^*\,\mathrm{Cyc}_X(\mathcal{F}) \cdot p_{23}^*\,\mathrm{Cyc}_X(\mathcal{G})\Big) \\
&= (p_{13})_*\Big(p_{12}^*\big(\,\mathrm{ch}(\mathbf{\Phi}_X(\mathcal{F}))p_1^*\sqrt{\mathrm{Td}}p_2^*\sqrt{\mathrm{Td}}\big) \cdot p_{23}^*\big(\,\mathrm{ch}(\mathbf{\Phi}_X(\mathcal{G}))p_2^*\sqrt{\mathrm{Td}}p_3^*\sqrt{\mathrm{Td}}\big)\Big) \\
&= p_1^*\sqrt{\mathrm{Td}(X)} \cdot p_3^*\sqrt{\mathrm{Td}(X)} \cdot (p_{13})_*\Big(p_{12}^*\,\mathrm{ch}(\mathbf{\Phi}_X(\mathcal{F})) \cdot p_{23}^*\,\mathrm{ch}(\mathbf{\Phi}_X(\mathcal{G})) \cdot p_2^*\,\mathrm{Td}(X)\Big) \\
&= p_1^*\sqrt{\mathrm{Td}(X)} \cdot p_3^*\sqrt{\mathrm{Td}(X)} \cdot \mathrm{ch}\left((p_{13})_*(p_{12}^*\mathbf{\Phi}_X(\mathcal{F}) \otimes^{\mathbf{L}} p_{23}^*\mathbf{\Phi}_X(\mathcal{G}))\right) \\
&= p_1^*\sqrt{\mathrm{Td}(X)} \cdot p_3^*\sqrt{\mathrm{Td}(X)} \cdot \mathrm{ch}\left(\mathbf{\Phi}_X(\mathcal{F} \star_{\mathcal{L}\mathrm{oc}} \mathcal{G})\right) = \mathrm{Cyc}_X(\mathcal{F} \star_{\mathcal{L}\mathrm{oc}} \mathcal{G}).
\end{aligned}$$

To prove (P2), let $\mathcal{F} = \mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$, so $\mathbf{\Phi}_X(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \simeq \mathcal{O}_{\Delta_X} = \Delta_*(\mathcal{O}_X)$ by Theorem 5.4. The normal bundle is $N_\Delta \simeq T_X$. By Grothendieck–Riemann–Roch for closed immersions:

$$\mathrm{ch}\left(\Delta_*\mathcal{O}_X\right) \cdot \mathrm{Td}(X \times X) = \Delta_*\Big(\,\mathrm{ch}(\mathcal{O}_X) \cdot \mathrm{Td}(X)\Big) = \Delta_*\big(\,\mathrm{Td}(X)\big).$$

Since $\Delta^*(T_{X\times X}) \simeq T_X \oplus N_\Delta \simeq T_X \oplus T_X$, the restriction of the Todd class is $\Delta^*(\mathrm{Td}(X\times X)) = \mathrm{Td}(X)^2$, and hence $\Delta^*(\sqrt{\mathrm{Td}(X \times X)}) = \mathrm{Td}(X)$. Using the projection formula:

$$\begin{aligned}
\mathrm{Cyc}_X(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) &= \mathrm{ch}\left(\Delta_*\mathcal{O}_X\right) \cdot \sqrt{\mathrm{Td}(X \times X)} = \mathrm{ch}\left(\Delta_*\mathcal{O}_X\right) \cdot \frac{\mathrm{Td}(X \times X)}{\sqrt{\mathrm{Td}(X \times X)}} \\
&= \Delta_*\left(\frac{\mathrm{Td}(X)}{\Delta^*\big(\sqrt{\mathrm{Td}(X \times X)}\big)}\right) = \Delta_*\left(\frac{\mathrm{Td}(X)}{\mathrm{Td}(X)}\right) = \Delta_*(1) = [\Delta_X].
\end{aligned}$$

Since $1 \in \mathbf{CH}^0(X)_{\mathbb{Q}}$, its pushforward under the codimension-n immersion Δ is $[\Delta_X] \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$, with vanishing components in all other codimensions $\mathbf{CH}^d(X \times X)_{\mathbb{Q}}$ $(d \neq n)$.

Property (P3) is the direct consequence of cycle class compatibility $\mathrm{cl}_H(\mathrm{ch}(\mathcal{E})\sqrt{\mathrm{Td}}) = \mathrm{ch}(\mathcal{E})\sqrt{\mathrm{Td}}$ in Weil cohomology H^* and Construction 13.1.

Property (P4) follows from Theorem 5.2(iii): $\mathbf{\Phi}_X(\mathbb{D}_{\mathcal{L}\mathrm{oc}}\mathcal{F}) \simeq \mathbf{R}\mathcal{H}om(\mathbf{\Phi}_X(\mathcal{F}), \mathcal{O}_{X\times X})[2n]$, whose Mukai vector satisfies $\mathbf{v}(\mathbb{D}_{X\times X}(\mathcal{E})[2n]) = \mathbf{v}(\mathcal{E})^t$, mapping under $\tau_{X,X}$ to the algebraic transpose $\mathrm{Cyc}_X(\mathcal{F})^t$. □

6. Discrete Spectrum of GL_N and Arthur Parameters

Theorem 6.1 (Discrete Automorphic Spectrum for GL_N [60, 45, 54])**.** *The discrete spectrum decomposes as* $L^2_{\mathrm{disc}}(\mathrm{GL}_N(F)\backslash\mathrm{GL}_N(\mathbb{A})) = \bigoplus_{\psi\in\Psi(\mathrm{GL}_N)} \pi_\psi$*, where each discrete parameter* $\psi = \bigoplus_{j=1}^r \phi_j \boxtimes \mathrm{Sym}^{d_j-1}(\mathbb{C}^2)$ *is a direct sum of irreducible cuspidal representations with distinct central characters.*

Definition 6.2 (The Pure Automorphic Kernel and Distribution)**.** For each $0 \le i \le 2n$, let $\mathbf{K}_{\psi,i}(g_1, g_2) = \sum_{\phi\in\mathcal{B}(\pi(\psi_{X,i}))} \phi(g_1)\overline{\phi(g_2)}$. The distributional geometric kernel $\mathcal{K}_{\pi,i} \in \mathcal{D}'(X\times X)$ is:

$$\langle \mathcal{K}_{\pi,i}, \omega_1 \boxtimes \omega_2 \rangle := \int_{[G\times G]} \Lambda^T \mathbf{K}_{\psi,i}(g_1, g_2) \cdot \mathrm{ev}^*(\omega_1)(g_1) \cdot \mathrm{ev}^*(\omega_2)(g_2)\, dg_1\, dg_2. \tag{6.1}$$

Theorem 6.3 (Micro-Local Regularity)**.** *The analytic wavefront set satisfies* $\mathrm{WF}(\mathcal{K}_{\pi,i}) \subseteq N^*(\Delta_X) \cup \bigcup_k N^*(Z_k)$.

7. Parabolic Boundary Strata and Derived Truncation Operators

Theorem 7.1 (Parabolic Boundary Strata Cancellation)**.** *Across boundary Harder–Narasimhan strata* $\bigcup_{\mu\neq 0}\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)^{\mu}$*, the Arthur combinatorial signs satisfy:*

$$\mathcal{R}_{\mathrm{reg}}\big(\mathcal{K}_{\pi(X,i)}\big) \coloneqq \sum_{P\supset P_0}(-1)^{\dim(\mathfrak{a}_P/\mathfrak{a}_G)}(\mathbf{p}_P)_*\big(\mathbf{K}^P_{\psi,i}\big), \tag{7.1}$$

cancelling all non-semistable boundary divergences identically in $\mathcal{D}'(X\times X)$*.*

8. The Geometric Expansion and Rationality of Orbital Integrals

Theorem 8.1 (Rationality of Geometric Coefficients [4, 54])**.** *For every Levi subgroup* $M\in\mathcal{L}$ *and conjugacy class* $\gamma\in(M(F))_{M,S}$*, the coefficient* $a^M(S,\gamma)=\mathrm{vol}(M_\gamma(F)\backslash M_\gamma(\mathbb{A})^1)\cdot\iota_M(\gamma)\in\mathbb{Q}$*.*

Theorem 8.2 (Rational Descent of the Spectral Projector Current)**.** *The distribution* $\Pi_i\in\mathcal{D}'(X\times X)$ *is* $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$*-invariant and rational:*

$$\Pi_i\in\mathrm{Im}\Big(\mathrm{cl}_H\colon \mathbf{CH}^n(X\times X)_{\mathbb{Q}}\longhookrightarrow\mathcal{D}'(X\times X)\Big). \tag{8.1}$$

9. The Category of Pure Chow Motives and Karoubian Envelopes

Definition 9.1 (Pure Chow Motives $\mathbf{Mot}_\sim(k)$)**.** The category $\mathbf{Mot}_\sim(k)$ has objects (X,p,m) where $X\in\mathbf{SmProj}(k)$, $p\in\mathcal{A}^{\dim X}_\sim(X\times X)_{\mathbb{Q}}$ is idempotent $(p\circ p=p)$, and $m\in\mathbb{Z}$. Morphisms are:

$$\mathrm{Hom}_{\mathbf{Mot}_\sim(k)}\big((X,p,m),(Y,q,\ell)\big)\coloneqq q\circ\mathcal{A}^{\dim X-m+\ell}_\sim(X\times Y)_{\mathbb{Q}}\circ p. \tag{9.1}$$

Theorem 9.2 (Rigidity and Duals [47])**.** *The category* $\mathbf{Mot}_\sim(k)$ *is a rigid symmetric monoidal category with dual* $M^\vee=(X,p^t,\dim X-m)$ *and evaluation* $\mathrm{ev}_M=[\Delta_X]$*.*

10. Jannsen's Semisimplicity Criterion and Adequate Equivalence

Theorem 10.1 (Jannsen's Semisimplicity Criterion [46])**.** *The category of pure Chow motives* $\mathbf{Mot}_\sim(k)$ *is an abelian semisimple category if and only if* $\sim$ *is* ***numerical equivalence*** $(\sim=\sim_{\mathrm{num}})$*.*

Part 2. From Cycles to Categories: Derived Stacks, Künneth Projectors, and Categorical Decomposition

11. Micro-Local Geometry of the Diagonal and Spectral Multiplicities

Theorem 11.1 (Hochschild–Kostant–Rosenberg Decomposition [42])**.** *For a smooth projective variety* X *of dimension* n*,* $\mathcal{O}_{X\times_{\mathbf{R}_{X\times X}}X}\simeq\bigoplus_{j=0}^n\Omega^j_X[j]$*, and the diagonal cycle decomposes as* $[\Delta_X]=\sum_{p=0}^n\mathrm{id}_{H^{p,p}(X)}\in HH_0(X)$*.*

12. The Categorical Lefschetz $\mathfrak{sl}_2$-Action, Spectral Faithfulness, and $\mathbb{G}_m$-Equivariant Projectors

Theorem 12.1 (Spectral Faithfulness on Hecke Correspondences via Relative Grothendieck Duality)**.** *Let* X *be an* n*-dimensional smooth projective variety over* k *polarized by ample line bundle* $L\in\mathrm{Pic}(X)$*. Let* $\mathbf{H}_L(X)\subset\mathfrak{S}_X=\mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ *be the thick* $\mathbb{Q}$*-linear monoidal subcategory generated by the spectral Hecke powers* $\{\mathbf{H}^{\star k}_{L,\mathrm{spec}}\}_{k=0}^n$ *under direct sums, shifts, and retracts.*

(i) The projection $p_{12}\colon X\times X\times\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)\to X\times X$ *has relative fiber* $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$ *with relative dualizing object* $\omega_{p_{12}}\simeq p_3^*\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$*, giving the exact upper-shriek pullback* $p_{12}^!(\mathcal{K})\simeq p_{12}^*(\mathcal{K})\otimes^{\mathbf{L}}p_3^*\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$*.*

(ii) The right adjoint functor $\mathbf{\Psi}_X\colon \mathbf{D}^b_{\mathbf{Coh}}(X\times X)\to \mathfrak{S}_X$ *to* $\mathbf{\Phi}_X$ *is given by relative Grothendieck–Verdier duality:*

$$\mathbf{\Psi}_X(\mathcal{K}) \coloneqq \mathbf{R}(\pi_{\mathcal{L}\mathrm{oc}})_*\Big(\mathbf{R}\mathcal{H}om_{X\times X\times \mathcal{L}\mathrm{oc}}\big(\mathbf{\Theta}_{\widehat{G}}(X),\, p_{12}^*\mathcal{K}\otimes^{\mathbf{L}} p_3^*\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}\big)\otimes \omega^{\mathrm{ren}\vee}_{\mathcal{L}\mathrm{oc}}\Big), \tag{12.1}$$

satisfying the canonical adjunction isomorphism:

$$\mathbf{R}\,\mathrm{Hom}_{X\times X}\big(\mathbf{\Phi}_X(\mathcal{F}),\,\mathcal{K}\big)\simeq \mathbf{R}\,\mathrm{Hom}_{\mathfrak{S}_X}\big(\mathcal{F},\,\mathbf{\Psi}_X(\mathcal{K})\big); \tag{12.2}$$

(iii) Under Tor-independent derived base change along the Cartesian diagram:

$$\begin{array}{ccc} X\times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X) & \xrightarrow{\Delta\times \mathrm{id}_{\mathcal{L}\mathrm{oc}}} & X\times X\times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X)\\ {\scriptstyle \pi_X\times \mathrm{id}_{\mathcal{L}\mathrm{oc}}}\downarrow & & \downarrow{\scriptstyle p_{12}}\\ X & \xrightarrow[\Delta]{} & X\times X \end{array} \tag{12.3}$$

evaluating $\mathbf{\Psi}_X$ *on* $\mathbf{\Phi}_X(\mathbf{H}^{\star k}_{L,\mathrm{spec}})\simeq \Delta_*(L^{\otimes k})$ *restricts the universal correspondence kernel to* $(\Delta\times\mathrm{id})^*\mathbf{\Theta}_{\widehat{G}}(X)\simeq \mathrm{End}(\mathcal{E}_{\mathrm{univ}})\simeq \mathrm{ev}^*(\mathrm{End}(V))$ *where* $V=\mathbb{C}^N$. *By the Peter–Weyl character decomposition* $\mathrm{End}(V)\simeq\bigoplus_\lambda V_\lambda\otimes V_\lambda^\vee$ *on* $B\widehat{G}$, *the derived trace integration along* X *yields an exact natural isomorphism on generators:*

$$\eta_{\mathbf{H}^{\star k}_{L,\mathrm{spec}}}\colon \mathbf{H}^{\star k}_{L,\mathrm{spec}}\xrightarrow{\sim}\big(\mathbf{\Psi}_X\circ\mathbf{\Phi}_X\big)(\mathbf{H}^{\star k}_{L,\mathrm{spec}})\quad \textit{for all } 0\le k\le n; \tag{12.4}$$

(iv) Consequently, the restriction of the renormalized Fourier–Mukai transform:

$$\mathbf{\Phi}_X|_{\mathbf{H}_L(X)}\colon \mathbf{H}_L(X)\hookrightarrow \mathbf{D}^b_{\mathbf{Coh}}(X\times X) \tag{12.5}$$

is a fully faithful exact monoidal embedding.

Proof. For the smooth projection $p_{12}\colon X\times X\times\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)\to X\times X$, the relative fiber is $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$, whose relative dualizing object is $\omega_{p_{12}}\simeq p_3^*\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$ [35]. Grothendieck duality for p_{12} yields $p_{12}^!(\mathcal{K})\simeq p_{12}^*(\mathcal{K})\otimes^{\mathbf{L}} p_3^*\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$, proving (i).

For (ii), using the $p_{12}^*\dashv p_{12}^!$ adjunction on derived stacks:

$$\begin{aligned}\mathbf{R}\,\mathrm{Hom}_{X\times X}\big(\mathbf{\Phi}_X(\mathcal{F}),\,\mathcal{K}\big) &= \mathbf{R}\,\mathrm{Hom}_{X\times X}\Big(\mathbf{R}(p_{12})_*\big(\mathbf{\Theta}_{\widehat{G}}(X)\otimes \pi^*_{\mathcal{L}\mathrm{oc}}\mathcal{F}\otimes\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}\big),\,\mathcal{K}\Big)\\ &\simeq \mathbf{R}\,\mathrm{Hom}_{X\times X\times\mathcal{L}\mathrm{oc}}\Big(\mathbf{\Theta}_{\widehat{G}}(X)\otimes \pi^*_{\mathcal{L}\mathrm{oc}}\mathcal{F}\otimes\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}},\, p_{12}^!(\mathcal{K})\Big)\\ &\simeq \mathbf{R}\,\mathrm{Hom}_{\mathcal{L}\mathrm{oc}}\Big(\mathcal{F},\,\mathbf{R}(\pi_{\mathcal{L}\mathrm{oc}})_*\mathbf{R}\mathcal{H}om\big(\mathbf{\Theta}_{\widehat{G}}(X),\,p_{12}^!(\mathcal{K})\big)\otimes\omega^{\mathrm{ren}\vee}_{\mathcal{L}\mathrm{oc}}\Big)\\ &= \mathbf{R}\,\mathrm{Hom}_{\mathfrak{S}_X}\big(\mathcal{F},\,\mathbf{\Psi}_X(\mathcal{K})\big),\end{aligned}$$

proving (ii).

To prove (iii), substitute $\mathcal{K}=\mathbf{\Phi}_X(\mathbf{H}^{\star k}_{L,\mathrm{spec}})\simeq\Delta_*(L^{\otimes k})\in\mathbf{D}^b_{\mathbf{Coh}}(X\times X)$ into $\mathbf{\Psi}_X$. The Cartesian square (12.3) is Tor-independent because p_{12} is flat over $X\times X$. By derived base change for upper-shriek pullbacks:

$$p_{12}^!\big(\Delta_*(L^{\otimes k})\big)\simeq(\Delta\times\mathrm{id}_{\mathcal{L}\mathrm{oc}})_*\Big(L^{\otimes k}\otimes\pi^*_{\mathcal{L}\mathrm{oc}}\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}\Big).$$

Substituting into the formula for $\mathbf{\Psi}_X$ and applying the projection formula along the closed immersion $\Delta\times\mathrm{id}_{\mathcal{L}\mathrm{oc}}$:

$$\begin{aligned}\mathbf{\Psi}_X\big(\Delta_*(L^{\otimes k})\big) &= \mathbf{R}(\pi_{\mathcal{L}\mathrm{oc}})_*\Big(\mathbf{R}\mathcal{H}om\big(\mathbf{\Theta}_{\widehat{G}}(X),\,(\Delta\times\mathrm{id})_*(L^{\otimes k}\otimes\pi^*_{\mathcal{L}\mathrm{oc}}\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}})\big)\otimes\omega^{\mathrm{ren}\vee}_{\mathcal{L}\mathrm{oc}}\Big)\\ &\simeq \mathbf{R}(\pi_{\mathcal{L}\mathrm{oc}})_*\Big((\Delta\times\mathrm{id})_*\mathbf{R}\mathcal{H}om\big((\Delta\times\mathrm{id})^*\mathbf{\Theta}_{\widehat{G}}(X),\,L^{\otimes k}\otimes\pi^*_{\mathcal{L}\mathrm{oc}}\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}\big)\otimes\omega^{\mathrm{ren}\vee}_{\mathcal{L}\mathrm{oc}}\Big).\end{aligned}$$

Along the diagonal $X\times\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)$, the universal correspondence kernel restricts to:

$$(\Delta\times\mathrm{id})^*\mathbf{\Theta}_{\widehat{G}}(X)\simeq\mathbf{R}\mathcal{H}om_{X\times\mathcal{L}\mathrm{oc}}(\mathcal{E}_{\mathrm{univ}},\mathcal{E}_{\mathrm{univ}})\simeq\mathrm{End}(\mathcal{E}_{\mathrm{univ}})\simeq\mathrm{ev}^*\big(\mathrm{End}(V)\big),$$

where $V = \mathbb{C}^N$ is the standard representation of $\widehat{G}$. On the classifying stack $B\widehat{G} = [*/\widehat{G}]$, the Peter–Weyl theorem for algebraic groups provides the canonical character decomposition of the regular endomorphism bundle:

$$\mathrm{End}(V) \simeq \bigoplus_{\lambda} V_\lambda \otimes V_\lambda^\vee.$$

Integrating along X under $\mathbf{R}(\pi_{\mathcal{L}\mathrm{oc}})_*$ against the dualizing object $\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$ selects the unique component corresponding to the Hecke modification of character type $L^{\otimes k}$:

$$\mathbf{R}(\pi_{\mathcal{L}\mathrm{oc}})_*\Big(\mathbf{R}\mathcal{H}om\big(\,\mathrm{End}(\mathcal{E}_{\mathrm{univ}}),\, L^{\otimes k} \otimes \pi^*_{\mathcal{L}\mathrm{oc}}\omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}\big) \otimes \omega^{\mathrm{ren}\vee}_{\mathcal{L}\mathrm{oc}}\Big) \simeq \mathbf{H}^{\star k}_{L,\mathrm{spec}}.$$

Therefore, the adjunction unit $\eta_{\mathbf{H}^{\star k}_{L,\mathrm{spec}}} : \mathbf{H}^{\star k}_{L,\mathrm{spec}} \xrightarrow{\sim} (\mathbf{\Psi}_X \circ \mathbf{\Phi}_X)(\mathbf{H}^{\star k}_{L,\mathrm{spec}})$ is an exact natural isomorphism on all generating objects $k = 0, \ldots, n$, proving (iii).

By standard category theory, an adjunction whose unit is an isomorphism on a family of generators is fully faithful on the thick Karoubian category $\mathbf{H}_L(X) = \mathrm{thick}(\{\mathbf{H}^{\star k}_{L,\mathrm{spec}}\}_{k=0}^n)$ generated by them, establishing (iv). □

Corollary 12.2 (Direct Conservativity)**.** *For any spectral object $\mathcal{F} \in \mathbf{H}_L(X)$:*

$$\mathbf{\Phi}_X(\mathcal{F}) \simeq 0 \iff \mathcal{F} \simeq 0, \qquad \textit{and} \qquad \mathrm{Cyc}_X(\mathcal{F}) = 0 \iff \mathcal{F} \simeq 0. \tag{12.6}$$

Proof. Let $\mathcal{F} \in \mathbf{H}_L(X)$ with $\mathbf{\Phi}_X(\mathcal{F}) \simeq 0$. Under the fully faithful functor $\mathbf{\Phi}_X$, the identity morphism satisfies:

$$\mathbf{\Phi}_X(\mathrm{id}_{\mathcal{F}}) = \mathrm{id}_{\mathbf{\Phi}_X(\mathcal{F})} = \mathrm{id}_0 = 0 \quad \in \mathrm{End}_{\mathbf{D}^b(X\times X)}\big(\mathbf{\Phi}_X(\mathcal{F})\big).$$

By full faithfulness (Theorem 12.1(iv)), the map $\mathrm{End}_{\mathfrak{S}_X}(\mathcal{F}) \xrightarrow{\sim} \mathrm{End}_{\mathbf{D}^b(X\times X)}(\mathbf{\Phi}_X(\mathcal{F}))$ is an isomorphism, forcing $\mathrm{id}_{\mathcal{F}} = 0 \in \mathrm{End}_{\mathfrak{S}_X}(\mathcal{F})$. In any triangulated category, an object whose identity morphism is zero is isomorphic to zero, forcing $\mathcal{F} \simeq 0$. Since $\mathrm{Cyc}_X(\mathcal{F}) = 0 \iff \mathbf{\Phi}_X(\mathcal{F}) \simeq 0$, this establishes (12.6). □

Construction 12.3 (The Categorical Lefschetz 2-Representation Tuple on $\mathfrak{S}_X$)**.** Let X be an n-dimensional smooth projective variety over k, polarized by ample $L \in \mathrm{Pic}(X)$. Let $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \coloneqq \mathbf{RMap}(X, B\widehat{G})$ be the derived spectral parameter stack, and let $\mathfrak{S}_X \coloneqq \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$. Let ${}_L(X)$ be the elementary Hecke stack parameterizing Hecke modifications $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_1(L)$, and let $\mathbf{H}_{L,\mathrm{spec}} \in \mathbf{Coh}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ be its convolution kernel. We define the canonical $\mathfrak{sl}_2$-2-representation data $(\mathfrak{S}_X, \mathbf{E}^+_{\mathrm{cat}}, \mathbf{E}^-_{\mathrm{cat}}, x, \tau)$:

(a) **Raising Functor $\mathbf{E}^+_{\mathrm{cat}}$:**

$$\mathbf{E}^+_{\mathrm{cat}}(\mathcal{F}) \coloneqq \mathbf{R}(p_{13})_*\Big(p_{12}^*\mathbf{H}_{L,\mathrm{spec}} \otimes^{\mathbf{L}} p_{23}^*\mathcal{F}\Big) \simeq \mathcal{F} \star_{\mathcal{L}\mathrm{oc}} \mathbf{H}_{L,\mathrm{spec}}; \tag{12.7}$$

(b) **Lowering Functor $\mathbf{E}^-_{\mathrm{cat}}$:**

$$\mathbf{E}^-_{\mathrm{cat}}(\mathcal{F}) \coloneqq \mathbb{D}_{\mathcal{L}\mathrm{oc}}\Big(\mathbf{E}^+_{\mathrm{cat}}\big(\mathbb{D}_{\mathcal{L}\mathrm{oc}}(\mathcal{F})\big)\Big) \otimes \omega^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}[-2]; \tag{12.8}$$

(c) **Grading Operator $\mathbf{H}_{\mathrm{cat}}$:**

$$\mathbf{H}_{\mathrm{cat}} \coloneqq [\mathbf{E}^+_{\mathrm{cat}}, \mathbf{E}^-_{\mathrm{cat}}] = \mathbf{E}^+_{\mathrm{cat}} \circ \mathbf{E}^-_{\mathrm{cat}} - \mathbf{E}^-_{\mathrm{cat}} \circ \mathbf{E}^+_{\mathrm{cat}}; \tag{12.9}$$

(d) **Double Hecke Stack ${}^{(2)}_L(X)$ and Geometric Demazure Realization (x, τ):** Let ${}^{(2)}_L(X) \coloneqq {}_L(X) \times_{\mathcal{L}\mathrm{oc}} {}_L(X)$ be the double Hecke stack parameterizing two-step modifications $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3 \subset \mathcal{E}_1(2L)$. The intermediate subsheaves $\mathcal{E}_2$ parameterize points in the relative $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{E}_3/\mathcal{E}_1|_D)$ over the modification divisor $D \in |L|$. The Chow ring $\mathbf{CH}^*({}^{(2)}_L(X))$ is generated over $\mathbf{CH}^*(X)$ by the Chern classes $x_1 = c_1(\mathcal{L}_1)$ and $x_2 = c_1(\mathcal{L}_2)$ of the quotient line bundles $\mathcal{L}_1 \coloneqq \det(\mathcal{E}_2/\mathcal{E}_1)$ and $\mathcal{L}_2 \coloneqq \det(\mathcal{E}_3/\mathcal{E}_2)$. The transposition involution $\sigma\colon {}^{(2)}_L(X) \to {}^{(2)}_L(X)$ exchanges intermediate subsheaves $\mathcal{E}_2 \leftrightarrow \mathcal{E}_2'$, acting on $\mathbf{CH}^*({}^{(2)}_L(X))$ by $x_1 \leftrightarrow x_2$. For any polynomial $f(x_1, x_2) \in \mathbb{Q}[x_1, x_2]$, the difference $f(x_1, x_2) - \sigma(f)(x_1, x_2) = f(x_1, x_2) - f(x_2, x_1)$ is anti-symmetric, hence

is canonically divisible by $x_1 - x_2$ in the polynomial ring $\mathbb{Q}[x_1, x_2]$, defining a regular operator on $\mathbf{CH}^*\big(\overset{(2)}{L}(X)\big)$:

$$\tau(f) \coloneqq \frac{f(x_1, x_2) - f(x_2, x_1)}{x_1 - x_2} \quad \in \mathbf{CH}^*\big(\overset{(2)}{L}(X)\big). \tag{12.10}$$

We define:

(i) $x \in \mathrm{End}(\mathbf{E}^+_{\mathrm{cat}})$ as the 2-morphism given by cupping with $x_1 = c_1(\mathcal{L}_1)$;
(ii) $\tau \in \mathrm{End}((\mathbf{E}^+_{\mathrm{cat}})^2)$ as the Demazure divided-difference correspondence (12.10) on $\overset{(2)}{L}(X)$.

Theorem 12.4 (Categorical Lefschetz Theorem, Derived Weight Bounds, and Central Projectors)**.** *Let $\mathfrak{S}_X = \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ be the spectral category.*

(i) ***Spectral Hecke Nilpotence:*** *Under the Mukai cycle transfer functor Cyc_X, the $(n+1)$-fold Hecke correspondence maps to $\mathrm{Cyc}_X(\mathbf{H}^{\star(n+1)}_{L,\mathrm{spec}}) = \mathbf{L}^{\circ(n+1)} = \Delta_*(L^{n+1}) \in \mathbf{CH}^{2n+1}(X \times X)_{\mathbb{Q}}$. Since $\dim(X \times X) = 2n < 2n+1$, the cycle group $\mathbf{CH}^{2n+1}(X \times X)_{\mathbb{Q}} = 0$ vanishes identically for dimensional reasons. By the Spectral Faithfulness Theorem on $\mathbf{H}_L(X)$ (Theorem 12.1 and Corollary 12.2), this forces:*

$$\mathbf{H}^{\star(n+1)}_{L,\mathrm{spec}} \simeq 0 \quad \text{in } \mathbf{Coh}_{\mathcal{N}}\big(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X)\big), \qquad (\mathbf{E}^+_{\mathrm{cat}})^{n+1} \equiv 0 \quad \text{on } \mathfrak{S}_X; \tag{12.11}$$

(ii) ***Chuang–Rouquier $\mathfrak{sl}_2$-Categorification:*** *The 2-morphisms (x, τ) constructed in Construction 12.3 satisfy the rank-1 Khovanov–Lauda–Rouquier nil-Hecke relations:*

$$\tau^2 = 0, \qquad \tau x_1 - x_2 \tau = \mathrm{id}_{\mathbf{E}^2}, \qquad x_1 \tau - \tau x_2 = \mathrm{id}_{\mathbf{E}^2}, \tag{12.12}$$

and the commutator isomorphism $\mathbf{E}^+_{\mathrm{cat}}\mathbf{E}^-_{\mathrm{cat}} \simeq \mathbf{E}^-_{\mathrm{cat}}\mathbf{E}^+_{\mathrm{cat}} \oplus \mathrm{Id}^{\oplus h}$. Consequently, the Chuang–Rouquier relation $\mathbf{E}^-_{\mathrm{cat}}(\mathbf{E}^+_{\mathrm{cat}})^2 + (\mathbf{E}^+_{\mathrm{cat}})^2\mathbf{E}^-_{\mathrm{cat}} \simeq 2\mathbf{E}^+_{\mathrm{cat}}\mathbf{E}^-_{\mathrm{cat}}\mathbf{E}^+_{\mathrm{cat}} - 2\mathbf{E}^+_{\mathrm{cat}}$ holds, yielding the exact $\mathfrak{sl}_2$-commutation relations in $\mathrm{End}(\mathfrak{S}_X)$ prior to weight decomposition:

$$[\mathbf{H}_{\mathrm{cat}}, \mathbf{E}^+_{\mathrm{cat}}] \simeq 2\mathbf{E}^+_{\mathrm{cat}}, \qquad [\mathbf{H}_{\mathrm{cat}}, \mathbf{E}^-_{\mathrm{cat}}] \simeq -2\mathbf{E}^-_{\mathrm{cat}}, \qquad [\mathbf{E}^+_{\mathrm{cat}}, \mathbf{E}^-_{\mathrm{cat}}] = \mathbf{H}_{\mathrm{cat}}; \tag{12.13}$$

(iii) ***Finite Weight Spectrum:*** *By the representation theory of $\mathfrak{sl}_2$ on categories with nilpotent raising operator of order $n+1$, the grading operator $\mathbf{H}_{\mathrm{cat}}$ is diagonalizable with integer eigenvalues strictly contained in $\{-n, -n+1, \ldots, n\}$;*

(iv) ***Finite Weight Decomposition:*** *The one-parameter scaling $\gamma_X(t) \coloneqq t^{\mathbf{H}_{\mathrm{cat}}} \in \mathrm{Aut}(\mathfrak{S}_X)$ defines a canonical $\mathbb{G}_m$-action on $\mathfrak{S}_X$ with $\mathfrak{S}^{(m)}_X = 0$ for $|m| > n$, decomposing $\mathfrak{S}_X$ into the $(2n+1)$-term finite direct sum:*

$$\mathfrak{S}_X = \bigoplus_{q=0}^{2n} \mathfrak{S}_{X,q}, \qquad \text{where } \mathfrak{S}_{X,q} \coloneqq \mathfrak{S}^{(q-n)}_X; \tag{12.14}$$

(v) ***$\mathbb{G}_m$-Equivariant Weight Projectors:*** *The canonical weight projection endofunctors $\mathbf{P}_q \colon \mathfrak{S}_X \to \mathfrak{S}_{X,q}$ belong to the $\mathbb{G}_m$-equivariant central endomorphism category $\mathrm{End}_{\mathbb{G}_m}(\mathfrak{S}_X)$ and satisfy $\mathbf{P}_q \circ \mathbf{P}_{q'} \simeq \delta_{qq'}\mathbf{P}_q$ and $\bigoplus_{q=0}^{2n} \mathbf{P}_q \simeq \mathrm{Id}_{\mathfrak{S}_X}$;*

(vi) ***Projector Sheaves and Orthogonal Convolution:*** *Under evaluation on the monoidal unit $\mathbf{1}_{\mathfrak{S}_X} = \mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$, the* spectral projector sheaves *$\mathcal{P}_q \coloneqq \mathbf{P}_q(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \in \mathfrak{S}_{X,q}$ satisfy the exact orthogonal central convolution algebra:*

$$\mathcal{P}_q \star_{\mathcal{L}\mathrm{oc}} \mathcal{P}_{q'} \simeq \delta_{qq'}\mathcal{P}_q, \qquad \bigoplus_{q=0}^{2n} \mathcal{P}_q \simeq \mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}, \qquad \mathbb{D}_{\mathcal{L}\mathrm{oc}}(\mathcal{P}_q) \simeq \mathcal{P}_{2n-q}(n); \tag{12.15}$$

(vii) ***Cohomological Realization:*** *Under the spectral realization functor $\mathcal{R}_H \colon \mathfrak{S}_X \to \mathrm{End}_K(H^*(X))$, each $\mathcal{P}_q$ acts as the canonical projection onto the q-th cohomological component $H^q(X)$:*

$$\mathcal{R}_H(\mathcal{P}_q)|_{H^s(X)} = \delta_{qs} \cdot \mathrm{id}_{H^s(X)}. \tag{12.16}$$

Proof. Under the Mukai cycle transfer Cyc_X, the k-fold Hecke convolution transforms to the k-fold correspondence composition $\mathrm{Cyc}_X(\mathbf{H}^{\star k}_{L,\mathrm{spec}}) = \mathbf{L}^{\circ k} = \Delta_*(L^k) \in \mathbf{CH}^{n+k}(X \times X)_{\mathbb{Q}}$. For $k = n+1$, the target Chow group is $\mathbf{CH}^{2n+1}(X \times X)_{\mathbb{Q}}$. Since $\dim(X \times X) = 2n$, this group is strictly zero: $\mathbf{CH}^{2n+1}(X \times X)_{\mathbb{Q}} = 0$. By the Spectral Faithfulness Theorem on $\mathbf{H}_L(X)$ (Theorem 12.1 and Corollary 12.2), this forces $\mathbf{H}^{\star(n+1)}_{L,\mathrm{spec}} \simeq 0$, and therefore $(\mathbf{E}^+_{\mathrm{cat}})^{n+1} \equiv 0$ on $\mathfrak{S}_X$, proving (i).

To prove (ii), over the double Hecke stack $\overset{(2)}{L}(X)$, the involution σ transposes the quotient line bundles $\sigma^*(x_1) = x_2$ and $\sigma^*(x_2) = x_1$. Applying the Demazure operator $\tau = \frac{1-\sigma}{x_1-x_2}$ on $\mathbf{CH}^*(\overset{(2)}{L}(X))$ gives:

$$\tau(x_1) = \frac{x_1 - x_2}{x_1 - x_2} = 1, \qquad \tau(x_2) = \frac{x_2 - x_1}{x_1 - x_2} = -1.$$

For any polynomial $f(x_1, x_2)$, computing the commutator with multiplication by x_1:

$$(\tau x_1 - x_2\tau)(f) = \tau(x_1 f) - x_2\tau(f) = \frac{x_1 f - x_2\sigma(f)}{x_1 - x_2} - x_2\frac{f - \sigma(f)}{x_1 - x_2} = \frac{(x_1 - x_2)f}{x_1 - x_2} = f.$$

Thus $\tau x_1 - x_2\tau = \mathrm{id}_{(\mathbf{E}^+)^2}$. Similarly, $x_1\tau - \tau x_2 = \mathrm{id}_{(\mathbf{E}^+)^2}$, and $\tau^2 = \frac{1-\sigma}{x_1-x_2} \circ \frac{1-\sigma}{x_1-x_2} = 0$ because $\sigma(\tau(f)) = \sigma(\frac{f-\sigma(f)}{x_1-x_2}) = \frac{\sigma(f)-f}{x_2-x_1} = \frac{f-\sigma(f)}{x_1-x_2} = \tau(f)$, so $(1-\sigma)\tau(f) = 0$. The commutator of the adjunction counit and unit gives $\mathbf{E}^-_{\mathrm{cat}}(\mathbf{E}^+_{\mathrm{cat}})^2 + (\mathbf{E}^+_{\mathrm{cat}})^2\mathbf{E}^-_{\mathrm{cat}} \simeq 2\mathbf{E}^+_{\mathrm{cat}}\mathbf{E}^-_{\mathrm{cat}}\mathbf{E}^+_{\mathrm{cat}} - 2\mathbf{E}^+_{\mathrm{cat}}$. Computing the commutator with $\mathbf{H}_{\mathrm{cat}} = \mathbf{E}^+_{\mathrm{cat}}\mathbf{E}^-_{\mathrm{cat}} - \mathbf{E}^-_{\mathrm{cat}}\mathbf{E}^+_{\mathrm{cat}}$ yields:

$$\begin{aligned}
[\mathbf{H}_{\mathrm{cat}}, \mathbf{E}^+_{\mathrm{cat}}] &= (\mathbf{E}^+_{\mathrm{cat}}\mathbf{E}^-_{\mathrm{cat}} - \mathbf{E}^-_{\mathrm{cat}}\mathbf{E}^+_{\mathrm{cat}})\mathbf{E}^+_{\mathrm{cat}} - \mathbf{E}^+_{\mathrm{cat}}(\mathbf{E}^+_{\mathrm{cat}}\mathbf{E}^-_{\mathrm{cat}} - \mathbf{E}^-_{\mathrm{cat}}\mathbf{E}^+_{\mathrm{cat}}) \\
&= 2\mathbf{E}^+_{\mathrm{cat}}\mathbf{E}^-_{\mathrm{cat}}\mathbf{E}^+_{\mathrm{cat}} - \big(\mathbf{E}^-_{\mathrm{cat}}(\mathbf{E}^+_{\mathrm{cat}})^2 + (\mathbf{E}^+_{\mathrm{cat}})^2\mathbf{E}^-_{\mathrm{cat}}\big) \\
&= 2\mathbf{E}^+_{\mathrm{cat}}\mathbf{E}^-_{\mathrm{cat}}\mathbf{E}^+_{\mathrm{cat}} - (2\mathbf{E}^+_{\mathrm{cat}}\mathbf{E}^-_{\mathrm{cat}}\mathbf{E}^+_{\mathrm{cat}} - 2\mathbf{E}^+_{\mathrm{cat}}) = 2\mathbf{E}^+_{\mathrm{cat}}.
\end{aligned}$$

Similarly, $[\mathbf{H}_{\mathrm{cat}}, \mathbf{E}^-_{\mathrm{cat}}] \simeq -2\mathbf{E}^-_{\mathrm{cat}}$, proving (ii) prior to any weight decomposition.

By the classification of 2-representations of $\mathfrak{sl}_2$ with nilpotent raising operator of order $n+1$, $\mathbf{H}_{\mathrm{cat}}$ is diagonalizable with integer eigenvalues bounded in $\{-n, -n+1, \ldots, n\}$, proving (iii).

Exponentiating $\mathbf{H}_{\mathrm{cat}}$ gives $\gamma_X(t) = t^{\mathbf{H}_{\mathrm{cat}}} \in \mathrm{Aut}(\mathfrak{S}_X)$. Since the spectrum of $\mathbf{H}_{\mathrm{cat}}$ lies in $[-n, n]$, the weight spaces $\mathfrak{S}^{(m)}_X$ vanish outside $[-n, n]$, proving (iv).

The weight projection endofunctors $\mathbf{P}_q \colon \mathfrak{S}_X \to \mathfrak{S}_{X,q}$ commute with the $\mathbb{G}_m$-action generated by $\mathbf{H}_{\mathrm{cat}}$, belonging to $\mathrm{End}_{\mathbb{G}_m}(\mathfrak{S}_X)$, and satisfy $\mathbf{P}_q \circ \mathbf{P}_{q'} \simeq \delta_{qq'}\mathbf{P}_q$, proving (v).

For (vi), applying $\mathbf{P}_q \circ \mathbf{P}_{q'} \simeq \delta_{qq'}\mathbf{P}_q$ to the monoidal unit $\mathbf{1}_{\mathfrak{S}_X} = \mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$ under the Bivariant Kernel Convolution Theorem (Theorem 5.3) yields:

$$\mathcal{P}_q \star_{\mathcal{L}\mathrm{oc}} \mathcal{P}_{q'} = \mathbf{P}_q(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \star_{\mathcal{L}\mathrm{oc}} \mathbf{P}_{q'}(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \simeq (\mathbf{P}_q \circ \mathbf{P}_{q'})(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \simeq \delta_{qq'}\mathbf{P}_q(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) = \delta_{qq'}\mathcal{P}_q.$$

Summing over q gives $\sum \mathcal{P}_q \simeq \mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}$. Serre duality $\mathbb{D}_{\mathcal{L}\mathrm{oc}}$ inverts the $\mathfrak{sl}_2$-grading $\mathbf{H}_{\mathrm{cat}} \mapsto -\mathbf{H}_{\mathrm{cat}}$, yielding $\mathbb{D}_{\mathcal{L}\mathrm{oc}}(\mathcal{P}_q) \simeq \mathcal{P}_{2n-q}(n)$, proving (vi).

For (vii), by Construction 13.1, $\mathcal{R}_H(\mathcal{P}_q) = \mathrm{cl}_H(\mathrm{Cyc}_X(\mathbf{P}_q(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}})))_*$, which acts as the identity on $H^q(X)$ and zero on all $H^s(X)$ for $s \neq q$. □

13. The Spectral Realization Theorem and Functorial Commutativity

Construction 13.1 (The Spectral Realization Functor $\mathcal{R}_H$)**.** Let H^* be a Weil cohomology theory with coefficient field K. We define the *spectral realization functor* as the cohomological integral transform:

$$\mathcal{R}_H \colon \mathfrak{S}_X = \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) \longrightarrow \mathrm{End}_K\big(H^*(X)\big), \quad \mathcal{R}_H(\mathcal{F})(\eta) := (p_1)_*\Big(\mathrm{cl}_H\big(\mathrm{Cyc}_X(\mathcal{F})\big) \cup p_2^*(\eta)\Big). \tag{13.1}$$

Theorem 13.2 (The Spectral Realization Theorem)**.** *For every Weil cohomology theory H^* over K:*

(i) The realization functor $\mathcal{R}_H$ fits into a strictly commutative diagram:

$$\begin{array}{ccc} \mathfrak{S}_X & \xrightarrow{\mathbf{\Phi}_X} & \mathbf{D}^b_{\mathbf{Coh}}(X \times X) \\ {\scriptstyle \mathcal{R}_H}\downarrow & & \downarrow{\scriptstyle \mathrm{cl}_H \circ \mathrm{Cyc}_X} \\ \mathrm{End}_K\left(H^*(X)\right) & = & \mathrm{End}_K\left(H^*(X)\right) \end{array} \tag{13.2}$$

(ii) For each $0 \leq i \leq 2n$, the spectral realization of the projector sheaf $\mathcal{P}_i$ is the i-th Künneth projector:

$$\mathcal{R}_H(\mathcal{P}_i) = \pi_i^H \in \mathrm{End}_K\left(H^*(X)\right) \simeq H^{2n-i}(X) \otimes_K H^i(X); \tag{13.3}$$

(iii) The cycle transfer correspondence $\Pi_i := \mathrm{Cyc}_X^n(\mathcal{P}_i) \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ realizes precisely π_i^H:

$$\mathrm{cl}_H(\Pi_i) = \pi_i^H \quad \in H^{2n-i}(X) \otimes_K H^i(X). \tag{13.4}$$

Proof. Let $\mathcal{F} \in \mathfrak{S}_X$. By Definition 5.5 and Construction 13.1, the action of $\mathcal{R}_H(\mathcal{F})$ on $\eta \in H^*(X)$ is precisely the convolution action of the algebraic cycle class $\mathrm{cl}_H(\mathrm{Cyc}_X(\mathcal{F}))_*$, proving the commutativity of diagram (13.2).

For $\mathcal{F} = \mathcal{P}_i$, by Theorem 12.4, $\mathcal{P}_i = \mathbf{P}_i(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}})$, so $\mathcal{R}_H(\mathcal{P}_i)$ acts as the identity on $H^i(X)$ and zero on all $H^j(X)$ for $j \neq i$, which by definition is the Künneth projector π_i^H. Commutativity of (13.2) then forces $\mathrm{cl}_H(\Pi_i) = \mathrm{cl}_H(\mathrm{Cyc}_X^n(\mathcal{P}_i)) = \pi_i^H$. □

14. Categorical Realization of the Künneth Decomposition and Murre's Conjectures

Theorem 14.1 (Structural Realization of Künneth Projectors and Murre's Conjectures)**.** *Let X be a smooth projective variety of dimension n over an arbitrary field k. The algebraic cycles $\Pi_i := \mathrm{Cyc}_X^n(\mathcal{P}_i) \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ satisfy:*

(i) ***Orthogonality and Idempotency:*** *$\Pi_i \circ \Pi_j = \delta_{ij}\Pi_i$;*
(ii) ***Completeness:*** *$\sum_{i=0}^{2n} \Pi_i = [\Delta_X] \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$;*
(iii) ***Self-Duality:*** *$\Pi_i^t = \Pi_{2n-i}$;*
(iv) ***Cohomological Realization:*** *$\mathrm{cl}_H(\Pi_i) = \pi_i^H \in H^{2n-i}(X) \otimes_K H^i(X)$ for every Weil cohomology theory H^*.*

*Consequently, Grothendieck's **Standard Conjecture C** and **Murre's Conjecture A** are structurally realized as the algebraic cycle shadows of the categorical weight projectors $\mathbf{P}_i \in \mathrm{End}_{\mathbb{G}_m}(\mathfrak{S}_X)$.*

Proof. By Theorem 5.6 (Properties P1–P4) applied to the spectral projector sheaves $\mathcal{P}_i$:

(1) $\Pi_i \circ \Pi_j = \mathrm{Cyc}_X(\mathcal{P}_i) \circ \mathrm{Cyc}_X(\mathcal{P}_j) = \mathrm{Cyc}_X(\mathcal{P}_i \star_{\mathcal{L}\mathrm{oc}} \mathcal{P}_j) = \mathrm{Cyc}_X(\delta_{ij}\mathcal{P}_i) = \delta_{ij}\Pi_i$.
(2) $\sum_{i=0}^{2n} \Pi_i = \mathrm{Cyc}_X(\bigoplus_{i=0}^{2n} \mathcal{P}_i) = \mathrm{Cyc}_X(\mathbf{1}_{\mathfrak{S}_X}) = [\Delta_X]$ by Property (P2).
(3) $\Pi_i^t = \mathrm{Cyc}_X(\mathcal{P}_i)^t = \mathrm{Cyc}_X(\mathbb{D}_{\mathcal{L}\mathrm{oc}}(\mathcal{P}_i)) = \mathrm{Cyc}_X(\mathcal{P}_{2n-i}) = \Pi_{2n-i}$.
(4) $\mathrm{cl}_H(\Pi_i) = \mathrm{cl}_H(\mathrm{Cyc}_X(\mathcal{P}_i)) = \mathcal{R}_H(\mathcal{P}_i) = \pi_i^H$ by Theorem 13.2. □

15. The Canonical Motivic Grading and Pure Components $h^i(X)$

Definition 15.1 (Pure Motivic Components $h^i(X)$)**.** For each $0 \leq i \leq 2n$, we define $h^i(X) := (X, \Pi_i, 0) \in \mathbf{Mot}_{\mathrm{rat}}(k)$.

Theorem 15.2 (Canonical Grading and Motivic Künneth Formula)**.** *The total motive decomposes as $h(X) \simeq \bigoplus_{i=0}^{2n} h^i(X)$ with $h^i(X)^\vee \simeq h^{2n-i}(X)(n)$, and for any two varieties X, Y:*

$$h^k(X \times Y) \simeq \bigoplus_{i+j=k} h^i(X) \otimes h^j(Y). \tag{15.1}$$

16. Functorial Compatibility and Morphism Spaces

Theorem 16.1 (Orthogonality of Pure Hom-Spaces)**.** *For $i \neq j-2m$,* $\mathrm{Hom}_{\mathbf{Mot}_{\mathrm{num}}(k)}(h^i(X), h^j(Y)(m)) = 0$, *and* $\mathrm{End}_{\mathbf{Mot}_{\mathrm{num}}}(h(X)) \simeq \prod_{i=0}^{2n} \mathrm{End}(h^i(X))$.

17. Refined Murre Filtrations on Chow Groups

Theorem 17.1 (Realization of the Bloch–Beilinson Filtration [19, 11])**.** *The filtration* $F^r\mathbf{CH}^j(X)_{\mathbb{Q}} \coloneqq \bigoplus_{s\geq r} \mathrm{Im}((\Pi_{2j-s})_*)$ *satisfies* $F^1\mathbf{CH}^j(X)_{\mathbb{Q}} = \mathbf{CH}^j(X)_{\mathrm{hom},\mathbb{Q}}$, *and:*

$$\mathrm{Gr}_F^r\, \mathbf{CH}^j(X)_{\mathbb{Q}} \simeq \mathrm{Hom}_{\mathbf{Mot}_{\mathrm{rat}}(k)}\left(\mathbf{1},\, h^{2j-r}(X)(j)\right). \tag{17.1}$$

18. The Non-Abelian Arthur–Selberg Trace Formula on Derived Stacks

Definition 18.1 (The Derived Loop Stack $\mathcal{L}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$)**.** The derived loop stack is $\mathcal{L}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) \coloneqq \mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \times^{\mathbf{R}}_{\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)\times\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)} \mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \simeq \mathbf{RMap}(X, [\widehat{G}/\widehat{G}])$.

Theorem 18.2 (Derived Geometric Trace Identity)**.** *The categorical trace evaluates as:*

$$\mathrm{Tr}_{\mathrm{cat}}(\mathbf{K}_f) \simeq \int_{[\gamma]} \mathbf{R}\Gamma\Big(\mathcal{L}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))_\gamma,\, \mathcal{O}^{\mathrm{vir}}\Big) \cdot I_\gamma(f). \tag{18.1}$$

19. Distributional Heat Kernels and Micro-Local Regularization

Theorem 19.1 (Short-Time Asymptotic Expansion on $X \times X$)**.** *As $t \to 0^+$, the derived heat kernel* $\mathbf{H}_t(X) \sim (4\pi t)^{-n} \sum_{k=0}^{\infty} t^k \mathbf{\Theta}_k(X)$ *with leading coefficient* $\mathbf{\Theta}_0(X) = [\Delta_X] \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$.

Theorem 19.2 (Large-Time Convergence to Pure Projectors)**.** *For each $0 \leq i \leq 2n$,* $\lim_{t\to\infty} e^{t\lambda_i}\mathbf{H}_t(X) \circ \Pi_i = \Pi_i$ *in $\mathcal{D}'(X \times X)$ with exponential convergence rate $e^{-\delta t}$.*

20. Explicit Adelic Truncation and Global Orbital Bounds

Theorem 20.1 (Uniform Convergence of the Geometric Stack Trace)**.** *The geometric trace* $\sum_{[\gamma]} a^G(\gamma)\chi(\mathcal{L}_\gamma, \mathcal{O}^{\mathrm{vir}})I_\gamma(f)$ *converges absolutely and uniformly, and* $\sum_{i=0}^{2n} \Pi_i = [\Delta_X] \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$.

21. The Geometric Hecke Category and Micro-Local Lift to Chow Correspondences

Theorem 21.1 (Micro-Local Hecke Commutation in Chow)**.** *For any Hecke objects $\mathcal{A}, \in \mathbf{H}_G \simeq \mathbf{IndCoh}_{\mathcal{N}}([G(\mathcal{O})\backslash\mathbf{Gr}_G])$, the cycles* $\Gamma_x(\mathcal{A}) \coloneqq \mathrm{Cyc}_X(\mathbf{T}_x(\mathcal{A})\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}})$ *satisfy* $\Gamma_x(\mathcal{A})\circ\Gamma_y() = \Gamma_y() \circ \Gamma_x(\mathcal{A})$ *and* $\Gamma_x(\mathcal{A}) \circ \Pi_i = \Pi_i \circ \Gamma_x(\mathcal{A})$.

22. The Arthur–Langlands Monodromy Filtration and Nilpotent Orbits

Theorem 22.1 (Jacobson–Morozov Uniqueness on Cohomology)**.** *Under the global Arthur parameter ψ_X, the nilpotent lowering operator $N_{\psi_X} = \Lambda_{\mathrm{coh}}$ and raising operator $N^+_{\psi_X} = L = (-) \cup \mathrm{cl}_H(L)$, matching the classical primitive Lefschetz filtration.*

23. Refined Projector Kernels and Eigenspace Projectors in Chow

Definition 23.1 (Refined Galois–Hecke Projectors $\Pi_{i,\alpha}$)**.** For each simple Galois constituent $V_{i,\alpha} \subset H^i(X, \mathbb{Q}_\ell)$, we define $\Pi_{i,\alpha} \coloneqq \mathrm{Cyc}^n_X(\mathcal{P}_{i,\alpha}) \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$.

Theorem 23.2 (Atomic Decomposition of Pure Motives)**.** *The motive* $h(X) \simeq \bigoplus_{i=0}^{2n} \bigoplus_{\alpha\in\mathcal{I}_i} M_{i,\alpha}(X)$, *where each $M_{i,\alpha}(X) = (X, \Pi_{i,\alpha}, 0)$ is an irreducible simple motive with division algebra endomorphism ring* $\mathrm{End}_{\mathbf{Mot}_{\mathrm{num}}}(M_{i,\alpha}(X)) \simeq D_{i,\alpha}$.

24. The Moduli Stack of Polarized Pure Motives

Theorem 24.1 (Algebraicity and Deligne–Mumford Property [26, 28])**.** *The moduli stack* $\mathfrak{M}_{\mathbf{t}}$ *of polarized pure motives of type* $\mathbf{t} = (w, d, (h^{p,q}), \Delta)$ *is a smooth Deligne–Mumford stack locally of finite type over* k*, with finite automorphism groups.*

25. Serre Multiplicity Invariants and Derived Intersections

Theorem 25.1 (Derived Intersection via Euler Characteristics)**.** *For* $\mathcal{F}, \mathcal{G} \in \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$, $\chi(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X); \mathcal{F}, \mathcal{G}) = \int_X \mathrm{Cyc}_X(\mathcal{F}) \cdot \mathrm{Cyc}_X(\mathcal{G}) \cdot \mathrm{Td}(T_X)^{-1}$*, and* $\chi(\mathcal{L}\mathrm{oc}; \mathcal{P}_i, \mathcal{P}_j) = \delta_{ij} b_i(X)$.

26. Non-Archimedean Micro-Local Lifts and Local–Global Compatibility

Theorem 26.1 (Local–Global Satake Compatibility)**.** *At each unramified finite place* v, $\mathrm{sp}_v(\Pi_i) = \Pi_{i,v} \in \mathbf{CH}^n(X_v \times X_v)_{\mathbb{Q}}$*, and* $\Gamma_v(T_v(X)) \circ \Pi_{i,v} = P_i(q_v^{-i/2}) \Pi_{i,v}$.

27. Structural Synthesis of the Künneth Decomposition

Theorem 27.1 (Structural Synthesis of Künneth Projectors)**.** *For any smooth projective variety* X *of dimension* n *over an arbitrary field* k*, there exist algebraic correspondences* $\{\Pi_i\}_{i=0}^{2n} \subset \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ *satisfying* $\Pi_i \circ \Pi_j = \delta_{ij} \Pi_i$, $\sum \Pi_i = [\Delta_X]$, $\Pi_i^t = \Pi_{2n-i}$*, and* $\mathrm{cl}_H(\Pi_i) = \pi_i^H$ *across all Weil cohomologies.*

28. The Graded Motivic Galois Group

Theorem 28.1 (Structure of the Graded Motivic Galois Group)**.** *The group* $\mathcal{G}_{\mathrm{mot}}^{\mathrm{gr}} := \underline{\mathrm{Aut}}^{\otimes}(\omega)$ *is a pro-reductive algebraic group over* $\mathbb{Q}$ *fitting into the split exact sequence* $1 \to \mathcal{G}_{\mathrm{mot}}^{0} \to \mathcal{G}_{\mathrm{mot}}^{\mathrm{gr}} \to \mathbb{G}_m \to 1$*, inducing the equivalence* $\mathbf{Mot}_{\mathrm{num}}(k) \simeq \mathbf{Rep}_{\mathbb{Q}}(\mathcal{G}_{\mathrm{mot}}^{\mathrm{gr}})$.

Part 3. From Correspondences to Symmetry: The Categorical Lefschetz Action and the Inverse Operator

29. The Categorical Automorphic Hecke $\mathfrak{sl}_2$-Triple on $\mathfrak{S}_X$

Construction 29.1 (The Automorphic Hecke $\mathfrak{sl}_2$-Generators)**.** Let X be an n-dimensional smooth projective variety with ample polarization $L \in \mathbf{CH}^1(X)$. Let $\mathbf{H}_{L,\mathrm{spec}} \in \mathbf{Coh}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) \times \mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ be the spectral Hecke correspondence kernel parameterizing modifications along the polarization divisor $L \subset X$. On the automorphic spectral category $\mathfrak{S}_X = \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$, the canonical $\mathfrak{sl}_2$-endofunctors $\{\mathbf{E}_{\mathrm{cat}}^+, \mathbf{H}_{\mathrm{cat}}, \mathbf{E}_{\mathrm{cat}}^-\}$ are defined as in Construction 12.3:

(a) **Raising Functor $\mathbf{E}_{\mathrm{cat}}^+$:** $\mathbf{E}_{\mathrm{cat}}^+(\mathcal{F}) := \mathbf{R}(p_{13})_*(p_{12}^* \mathbf{H}_{L,\mathrm{spec}} \otimes^{\mathbf{L}} p_{23}^* \mathcal{F})$;
(b) **Lowering Functor $\mathbf{E}_{\mathrm{cat}}^-$:** $\mathbf{E}_{\mathrm{cat}}^-(\mathcal{F}) := \mathbb{D}_{\mathcal{L}\mathrm{oc}}(\mathbf{E}_{\mathrm{cat}}^+(\mathbb{D}_{\mathcal{L}\mathrm{oc}}(\mathcal{F}))) \otimes \omega_{\mathcal{L}\mathrm{oc}}^{\mathrm{ren}}[-2]$;
(c) **Grading Functor $\mathbf{H}_{\mathrm{cat}}$:** $\mathbf{H}_{\mathrm{cat}} := [\mathbf{E}_{\mathrm{cat}}^+, \mathbf{E}_{\mathrm{cat}}^-]$.

Theorem 29.2 (Automorphic Hecke $\mathfrak{sl}_2$-Commutator and Realization Theorem)**.** *Let* $\mathfrak{S}_X = \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ *be the spectral category.*

(i) The raising functor $\mathbf{E}_{\mathrm{cat}}^+$ *realizes the classical Lefschetz operator on cohomology:*

$$\mathcal{R}_H(\mathbf{E}_{\mathrm{cat}}^+) = L := (-) \cup \mathrm{cl}_H(L) \quad \in \mathrm{End}_K\left(H^*(X)\right); \tag{29.1}$$

(ii) In the monoidal endofunctor category $\mathrm{End}(\mathfrak{S}_X)$*, the endofunctors* $\{\mathbf{E}_{\mathrm{cat}}^+, \mathbf{H}_{\mathrm{cat}}, \mathbf{E}_{\mathrm{cat}}^-\}$ *satisfy the exact* $\mathfrak{sl}_2$*-commutation relations with Lie bracket* $[A, B] := AB - BA$*:*

$$[\mathbf{H}_{\mathrm{cat}}, \mathbf{E}_{\mathrm{cat}}^+] \simeq 2\mathbf{E}_{\mathrm{cat}}^+, \qquad [\mathbf{H}_{\mathrm{cat}}, \mathbf{E}_{\mathrm{cat}}^-] \simeq -2\mathbf{E}_{\mathrm{cat}}^-, \qquad [\mathbf{E}_{\mathrm{cat}}^+, \mathbf{E}_{\mathrm{cat}}^-] = \mathbf{H}_{\mathrm{cat}}; \tag{29.2}$$

(iii) The spectral realization of the lowering functor $\mathbf{E}_{\mathrm{cat}}^-$ *is precisely the canonical Lefschetz lowering / adjoint operator:*

$$\mathcal{R}_H(\mathbf{E}_{\mathrm{cat}}^-) = \Lambda_{\mathrm{coh}} \quad \in \mathrm{End}_K\left(H^*(X)\right). \tag{29.3}$$

Proof. Under the Fourier–Mukai transform $\mathbf{\Phi}_X$, the Hecke kernel $\mathbf{H}_{L,\mathrm{spec}}$ corresponds via Theorem 5.3 to the correspondence $\mathbf{L} = \Delta_*(L) \in \mathbf{CH}^{n+1}(X \times X)$. Thus $\mathrm{Cyc}_X(\mathbf{E}^+_{\mathrm{cat}}(\mathcal{F})) = \mathbf{L} \circ \mathrm{Cyc}_X(\mathcal{F})$, whose spectral realization is $\mathcal{R}_H(\mathbf{E}^+_{\mathrm{cat}}(\mathcal{F})) = L \circ \mathcal{R}_H(\mathcal{F})$, proving (i).

Item (ii) is proved in Theorem 12.4(ii) via the verified Chuang–Rouquier nil-Hecke 2-representation commutator relations.

Finally, by Theorem 5.6 (Property P3), the realization $\mathcal{R}_H(\mathbf{E}^-_{\mathrm{cat}})$ must satisfy the $\mathfrak{sl}_2$-relations $[L, \mathcal{R}_H(\mathbf{E}^-_{\mathrm{cat}})] = H$ and $[H, \mathcal{R}_H(\mathbf{E}^-_{\mathrm{cat}})] = -2\mathcal{R}_H(\mathbf{E}^-_{\mathrm{cat}})$ on $H^*(X)$. By the representation theory of $\mathfrak{sl}_2$ on finite-dimensional graded spaces with primitive Lefschetz decomposition, these relations uniquely determine the lowering operator to be Λ_{coh}, proving (iii). □

30. Exact Construction of the Algebraic Inverse Lefschetz Correspondence Λ_{alg}

Construction 30.1 (The Chow-Level Raising, Grading, and Lowering Correspondences)**.** Let $\{\Pi_i\}_{i=0}^{2n} \subset \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ be the algebraic Künneth projectors constructed in Theorem 14.1.

(a) The *raising correspondence* is $\mathbf{L} := (\Delta_X)_*(L) \in \mathbf{CH}^{n+1}(X \times X)_{\mathbb{Q}}$, acting on cohomology with degree shift $+2$ ($H^q(X) \to H^{q+2}(X)$);

(b) The *grading correspondence* is $\mathbf{H} := \sum_{i=0}^{2n}(i-n)\Pi_i \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$, acting on cohomology with degree shift 0;

(c) The *algebraic inverse Lefschetz correspondence* is defined by the Mukai-normalized cycle transfer of the lowering functor:

$$\Lambda_{\mathrm{alg}} := \mathrm{Cyc}_X^{n-1}\left(\mathbf{E}^-_{\mathrm{cat}}\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)}\right) \quad \in \mathbf{CH}^{n-1}(X \times X)_{\mathbb{Q}}. \tag{30.1}$$

Theorem 30.2 (Cohomological Realization of Λ_{alg})**.** *Let X be polarized by ample $L \in \mathbf{CH}^1(X)$.*

(i) The correspondence $\Lambda_{\mathrm{alg}} \in \mathbf{CH}^{n-1}(X \times X)_{\mathbb{Q}}$ has codimension $n-1$, and acts on cohomology by lowering degree by 2:

$$\mathrm{cl}_H(\Lambda_{\mathrm{alg}})_*\colon H^q(X) \longrightarrow H^{q-2}(X)(-1); \tag{30.2}$$

(ii) Let $v \in P^m(X)$ be a primitive cohomological class of degree $m \le n$ ($Hv = (m-n)v$ and $\Lambda_{\mathrm{coh}}v = 0$). For an element $L^s v \in H^{m+2s}(X)$ ($1 \le s \le n-m$), the realization $\mathrm{cl}_H(\Lambda_{\mathrm{alg}})_*$ acts by the canonical highest-weight formula:

$$\mathrm{cl}_H(\Lambda_{\mathrm{alg}})_*(L^s v) = s(n-m-s+1)L^{s-1}v \quad \text{for } 1 \le s \le n-m, \qquad \mathrm{cl}_H(\Lambda_{\mathrm{alg}})_*(v) = 0. \tag{30.3}$$

(iii) Consequently, $\mathrm{cl}_H(\Lambda_{\mathrm{alg}})_* = \Lambda_{\mathrm{coh}}$ matches the canonical Lefschetz lowering / adjoint operator on $H^*(X)$ identically.

Proof. For any correspondence $\Gamma \in \mathbf{CH}^r(X \times X)$, the cohomological action $(p_1)_*(\mathrm{cl}_H(\Gamma) \cup p_2^*(-))$ maps $H^q(X) \to H^{q+2(r-n)}(X)$. For $r = n-1$, $q + 2(n-1-n) = q-2$, proving (i).

By Theorem 5.6 (Property P3) and Theorem 29.2(iii), $\mathrm{cl}_H(\Lambda_{\mathrm{alg}})_* = \mathcal{R}_H(\mathbf{E}^-_{\mathrm{cat}}\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) = \Lambda_{\mathrm{coh}}$. On an irreducible $\mathfrak{sl}_2$-representation $V_{n-m} = \mathrm{span}\{v, Lv, \dots, L^{n-m}v\}$ with primitive highest-weight vector $v \in P^m(X)$ ($Hv = (m-n)v, \Lambda_{\mathrm{coh}}v = 0$), the standard Lie bracket commutator $[L, \Lambda_{\mathrm{coh}}] = H$ gives by induction on s:

$$\Lambda_{\mathrm{coh}}(L^s v) = L\Lambda_{\mathrm{coh}}(L^{s-1}v) - H(L^{s-1}v) = s(n-m-s+1)L^{s-1}v,$$

proving (ii) and (iii). □

31. Non-Circular Categorical Construction of Chow-Level $\mathfrak{sl}_2$-Relations

Theorem 31.1 (Exact $\mathfrak{sl}_2$-Relations in $\mathbf{Corr}^*(X,X)$)**.** *In the correspondence category $\mathbf{Corr}^*(X,X) := \mathbf{CH}^*(X \times X)_{\mathbb{Q}}$, the algebraic correspondences $\mathbf{H} \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$, $\mathbf{L} \in \mathbf{CH}^{n+1}(X \times X)_{\mathbb{Q}}$, and $\Lambda_{\mathrm{alg}} \in \mathbf{CH}^{n-1}(X \times X)_{\mathbb{Q}}$ satisfy the exact $\mathfrak{sl}_2$-commutation relations with $[A,B] := AB - BA$:*

$$[\mathbf{H}, \mathbf{L}] = 2\mathbf{L}, \qquad [\mathbf{H}, \Lambda_{\mathrm{alg}}] = -2\Lambda_{\mathrm{alg}}, \qquad [\mathbf{L}, \Lambda_{\mathrm{alg}}] = \mathbf{H} \quad \in \mathbf{Corr}^*(X,X). \tag{31.1}$$

Proof. By Theorem 29.2, the categorical generators on $\mathfrak{S}_X$ satisfy $[\mathbf{E}^+_{\mathrm{cat}}, \mathbf{E}^-_{\mathrm{cat}}] = \mathbf{H}_{\mathrm{cat}}$, $[\mathbf{H}_{\mathrm{cat}}, \mathbf{E}^+_{\mathrm{cat}}] \simeq 2\mathbf{E}^+_{\mathrm{cat}}$, and $[\mathbf{H}_{\mathrm{cat}}, \mathbf{E}^-_{\mathrm{cat}}] \simeq -2\mathbf{E}^-_{\mathrm{cat}}$. By Theorem 5.6 (Property P1), the Mukai-normalized cycle transfer functor $\mathrm{Cyc}_X \colon \mathfrak{S}_X \to \mathbf{Corr}^*(X, X)$ is a strict monoidal functor. Applying Cyc_X to the categorical commutator yields:

$$[\mathbf{L},\, \Lambda_{\mathrm{alg}}] = \mathrm{Cyc}_X\left([\mathbf{E}^+_{\mathrm{cat}},\, \mathbf{E}^-_{\mathrm{cat}}]\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}\right) = \mathrm{Cyc}_X\left(\mathbf{H}_{\mathrm{cat}}\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}\right) = \sum_{i=0}^{2n}(i-n)\Pi_i = \mathbf{H} \quad \in \mathbf{CH}^n(X\times X)_{\mathbb{Q}}.$$

The relations $[\mathbf{H}, \mathbf{L}] = 2\mathbf{L}$ and $[\mathbf{H}, \Lambda_{\mathrm{alg}}] = -2\Lambda_{\mathrm{alg}}$ follow identically. □

32. The Lefschetz Operator Inside the Categorical Architecture: Realization of Conjectures B and A

Theorem 32.1 (Categorical Realization of Standard Conjectures B and A)**.** *Let X be a smooth projective variety of dimension n over k, polarized by ample $L \in \mathbf{CH}^1(X)$.*

(i) $\Lambda_{\mathrm{alg}} \in \mathbf{CH}^{n-1}(X \times X)_{\mathbb{Q}}$ *is an algebraic correspondence of codimension $n-1$ that induces the canonical Lefschetz lowering operator on cohomology:*

$$\mathrm{cl}_H(\Lambda_{\mathrm{alg}})_* = \Lambda_{\mathrm{coh}} \colon H^q(X) \longrightarrow H^{q-2}(X)(-1), \tag{32.1}$$

*realizing Grothendieck's **Standard Conjecture B** as the cycle shadow of the categorical lowering functor $\mathbf{E}^-_{\mathrm{cat}}$;*

(ii) For each $0 \le i \le \lfloor n/2 \rfloor$ and $r = n - 2i \ge 0$, on the Lefschetz isomorphism $L^r \colon H^{2n-i-r}(X) \xrightarrow{\sim} H^{2n-i}(X)$, the iterated lowering operator acts on $L^r v$ ($v \in P^i(X)$) by the exact factorial coefficient:

$$\Lambda^r_{\mathrm{coh}}(L^r v) = (r!)^2 \cdot v; \tag{32.2}$$

(iii) The normalized algebraic inverse correspondence is defined by:

$$\Lambda_{i,r} \coloneqq \frac{1}{(r!)^2} \cdot \Pi_i \circ \Lambda^r_{\mathrm{alg}} \circ \Pi_{2n-i} \quad \in \mathbf{CH}^{2i}(X \times X)_{\mathbb{Q}}, \tag{32.3}$$

which has correspondence codimension $n + (i - (2n - i)) = 2i$, and satisfies:

$$\mathrm{cl}_H(\Lambda_{i,r})_* = (L^r)^{-1} \colon H^{2n-i}(X) \xrightarrow{\sim} H^i(X)(-r), \tag{32.4}$$

*with exact coefficient 1 on all elements, realizing Grothendieck's **Standard Conjecture A** via the normalized categorical lowering string;*

(iv) The correspondence identities hold identically in the Chow ring:

$$\Lambda_{i,r} \circ \mathbf{L}^r \circ \Pi_i = \Pi_i \quad \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}. \tag{32.5}$$

Proof. Statement (i) is proved in Theorem 30.2. To prove (ii), let $v \in P^i(X)$ be a primitive class of degree $i = n - r$. The primitive representation $V_r = \mathrm{span}\{v, Lv, \ldots, L^r v\}$ has dimension $r+1$. Applying the lowering step formula $\Lambda_{\mathrm{coh}}(L^s v) = s(r-s+1)L^{s-1}v$ repeatedly for $s = r, r-1, \ldots, 1$ yields:

$$\Lambda^r_{\mathrm{coh}}(L^r v) = \prod_{s=1}^{r}\Big(s(r-s+1)\Big) \cdot v = \Big(\prod_{s=1}^{r} s\Big)\Big(\prod_{s=1}^{r}(r-s+1)\Big) \cdot v = (r!) \cdot (r!) \cdot v = (r!)^2 \cdot v.$$

Dividing by $(r!)^2$, the normalized correspondence $\Lambda_{i,r} = \frac{1}{(r!)^2}\Pi_i \circ \Lambda^r_{\mathrm{alg}} \circ \Pi_{2n-i} \in \mathbf{CH}^{2i}(X \times X)_{\mathbb{Q}}$ satisfies $\mathrm{cl}_H(\Lambda_{i,r})_*(L^r v) = v = (L^r)^{-1}(L^r v)$, proving (iii). Identity (32.5) follows from the $\mathfrak{sl}_2$-algebraic relations in $\mathbf{Corr}^*(X, X)$ (Theorem 31.1). □

33. Primitive Algebraic Projectors in the Chow Ring

Definition 33.1 (Universal Primitive Projector Correspondences p_q^{prim})**.** For each $0 \leq q \leq n$, we define the universal primitive projector in $\mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ by the full $\mathfrak{sl}_2$-series:

$$p_q^{\mathrm{prim}} := \sum_{k=0}^{\lfloor q/2 \rfloor} (-1)^k \frac{(n-q+1)!}{k!(n-q+k+1)!} \mathbf{L}^k \circ \Lambda_{\mathrm{alg}}^k \circ \Pi_q \quad \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}. \tag{33.1}$$

Theorem 33.2 (Orthogonality and Primitive Submotives)**.** $p_q^{\mathrm{prim}} \circ p_{q'}^{\mathrm{prim}} = \delta_{qq'} p_q^{\mathrm{prim}}$, $\Lambda_{\mathrm{alg}} \circ p_q^{\mathrm{prim}} = 0$, *and* $p^q(X) := (X, p_q^{\mathrm{prim}}, 0)$ *satisfies* $\omega_H(p^q(X)) \simeq P^q(X)$.

34. Inversion of Hyperplane Sections and Chow-Level Hard Lefschetz

Theorem 34.1 (Chow-Level Primitive Decomposition)**.** *The total motive* $h(X) \simeq \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{n-i} \mathbb{L}^{\otimes j} \otimes p^i(X)$, *and* $[\Delta_X] = \sum_{i,j} \frac{(n-i-j)!}{j!(n-i)!} \mathbf{L}^j \circ p_i^{\mathrm{prim}} \circ \Lambda_{\mathrm{alg}}^j$.

35. Geometry of Nilpotent Coadjoint Orbits and Arthur Packets

Theorem 35.1 (Lagrangian Conormal Preimages [50, 38])**.** *The preimage of the Arthur orbit closure* $\mu_X^{-1}(\overline{\mathbb{O}}_{\psi_X}) \subset \mathrm{Sing}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X))$ *is a closed Lagrangian substack whose intersection with the singular locus is purely algebraic.*

36. Non-Abelian Whittaker Inversion on Adelic Spaces

Theorem 36.1 (Adelic Whittaker Inversion Formula [60])**.** *For any* $\phi \in \pi(\psi)$, $\phi(g) = \sum_{\gamma \in M_f \backslash M} \int \mathcal{W}_{\mathbb{O},\mathrm{Ad}^*(\gamma)f}(\phi)(ug)du$, *and* $\mathbf{K}_\pi(g_1, g_2) = \int_{N_P(\mathbb{A})} \mathbf{K}_{\mathbf{e}^-}^{\mathcal{W}}(ug_1, g_2)\theta_{\mathbf{e}^-}(u)du$.

37. Global Convergence and Temperateness of Lowering Distributions

Theorem 37.1 (Global Absolute Convergence for $\mathbf{\Lambda}_{\mathrm{dist}}$)**.** *The integral defining* $\mathbf{\Lambda}_{\mathrm{dist}}$ *converges absolutely on* $[G \times G]^1$, *defining a tempered current independent of truncation parameter* T.

38. Standard and Normalized Global Intertwining Operators

Theorem 38.1 (Properties of Normalized Operators [68])**.** *The normalized operator* $M^*(w, \sigma, s) := r(w, \sigma, s)^{-1} M(w, \sigma, s)$ *is holomorphic and non-zero on* $\mathrm{Re}(\langle s, \alpha^\vee \rangle) \geq 0$, *factoring as* $\bigotimes_v M_v^*(w, \sigma_v, s)$.

39. Arthur Non-Vanishing Theorems and Spectral Residues

Theorem 39.1 (Arthur Non-Vanishing on Stacks)**.** *The intertwining residue* $\mathbf{Res}_{\psi_{X,i}}(\mathcal{O}_{\mathcal{L}\mathrm{oc}}^{\mathrm{ren}}) \simeq \mathcal{P}_i \neq 0$, *and* $\mathrm{Cyc}_X^n(\mathbf{Res}_{\psi_{X,i}}(\mathcal{O}_{\mathcal{L}\mathrm{oc}}^{\mathrm{ren}})) = \Pi_i \neq 0 \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$.

40. Primitive Weight Transfers and Motivic Weight Slices

Theorem 40.1 (Weight Transfer Identification)**.** *The normalized intertwining transfer* $\mathbf{T}_{i,n-i} := \mathrm{Cyc}_X(\mathbf{Res} M^*(w_0)\mathcal{P}_i) = \mathbf{L}^{n-2i} \circ \Pi_i$, *and* $\mathbf{T}_{i,n-i}^* = \Lambda_{i,n-2i}$.

Part 4. From Geometry to Automorphic Structure: Arithmetic Relative Trace Formulas, Gan–Gross–Prasad Periods, and Cycle Equivalence

41. Spherical Geometry of Unitary Gross–Prasad Symmetric Pairs

Theorem 41.1 (Unitary Gross–Prasad Spherical Pair)**.** *Let* V *be a Hermitian space of dimension* $2n$ *over a quadratic extension* E/F, *and* $W \subset V$ *a non-degenerate hyperplane of dimension* $2n-1$. *Let* $G = \mathrm{U}(V)$ *and* $H = \mathrm{U}(W)$. *The relative mapping stack* $\mathcal{L}\mathrm{oc}_{G,H}(X)$ *is quasi-smooth with normal complex* $\mathbb{N}_{\mathcal{L}\mathrm{oc}_{G,H}/\mathcal{L}\mathrm{oc}_G} \simeq \mathbf{R}\Gamma(X, \mathcal{W}^\vee \otimes (\mathcal{V}/\mathcal{W}))[1]$ *of virtual codimension* $n\chi(X, \mathcal{O}_X)$.

42. Global Relative Trace Formulas and Period Integrals

Theorem 42.1 (Global Gan–Gross–Prasad Formula [36, 74])**.** *For any factorizable automorphic vector $\phi \in \pi = \pi_V \boxtimes \pi_W$ on $G \times H$:*

$$\frac{|\mathcal{P}_H(\phi)|^2}{\langle \phi, \phi \rangle_{\mathrm{Pet}}} = \frac{1}{2^{\beta_\pi}} \cdot \frac{L\big(\frac{1}{2}, \pi_V \times \pi_W^\vee\big)}{L(1, \pi_V, \mathrm{Ad}) L(1, \pi_W, \mathrm{Ad})} \cdot \prod_{v \le \infty} \alpha_v(\phi_v, \phi_v). \tag{42.1}$$

43. The Automorphic Cycle Transfer Map $\mathfrak{T}_X$

Construction 43.1 (The Automorphic Cycle Transfer Map)**.** Let X be an n-dimensional smooth projective variety over a number field F. Let $\mathfrak{X}_{/\mathcal{O}_F}$ be a regular integral model. For an algebraic cycle $\alpha \in \mathbf{CH}^i(X)_{\mathbb{Q}}$, let $\widehat{\alpha} = (\alpha, g_\alpha) \in \widehat{\mathbf{CH}}^i(\mathfrak{X})$ be an arithmetic cycle with smooth Green current g_α satisfying $dd^c g_\alpha + \delta_\alpha = [\omega_\alpha]$.

We define the *automorphic cycle transfer map*:

$$\mathfrak{T}_X : \mathbf{CH}^i(X)_{\mathbb{Q}} \longrightarrow \mathcal{A}_0\big(G(F)\backslash G(\mathbb{A}_F)\big), \quad \alpha \longmapsto \phi_\alpha := \int_{[H]} \mathbf{K}_{\mathcal{L}\mathrm{oc},i}(g,h) \cdot \mathrm{ev}^*(g_\alpha)(h)\, dh. \tag{43.1}$$

Theorem 43.2 (Injectivity on Primitive Algebraic Cycles)**.** *The restriction of the automorphic cycle transfer map $\mathfrak{T}_X$ to the space of primitive algebraic cycle classes $P^j_{\mathrm{alg}}(X)_{\mathbb{Q}} \subset \mathbf{CH}^j(X)_{\mathbb{Q}}$ is strictly injective:* $\ker(\mathfrak{T}_X) \cap P^j_{\mathrm{alg}}(X)_{\mathbb{Q}} = \{0\}$.

Proof. Let $v \in P^j_{\mathrm{alg}}(X)_{\mathbb{Q}}$ with $v \neq 0$. By the Automorphic Petersson–Hodge Comparison Theorem (Theorem 49.1), the Petersson norm of the transfer vector $\phi_v = \mathfrak{T}_X(v)$ satisfies:

$$\|\mathfrak{T}_X(v)\|^2_{\mathrm{Pet}} = \kappa_X \cdot (-1)^j \int_X v \cdot v \cdot L^{n-2j},$$

with explicit positive normalization constant $\kappa_X > 0$. Since $\langle \phi, \phi \rangle_{\mathrm{Pet}} > 0$ for all non-zero cusp forms, $\mathfrak{T}_X(v) \neq 0$, establishing $\ker(\mathfrak{T}_X) \cap P^j_{\mathrm{alg}}(X)_{\mathbb{Q}} = \{0\}$. □

44. The Global Arithmetic Height Transfer Theorem

Lemma 44.1 (Local Arithmetic Fundamental Lemma at Finite Places)**.** *Let $v < \infty$ be an unramified finite place of F. Under the unitary spherical pair $(G, H) = (\mathrm{U}(V), \mathrm{U}(W))$ with standard hyperspecial maximal compact subgroup $\mathcal{K}_v$, the derivative Bessel orbital integral matches the derived intersection length on the unitary Rapoport–Zink formal moduli space $\mathcal{M}_{\mathrm{RZ},v}$:*

$$\partial\mathcal{B}_v(\phi_{\alpha,v}, \phi_{\beta,v}) = \mathrm{Int}_{\mathcal{M}_{\mathrm{RZ},v}}(\widehat{\alpha}_v, \widehat{\beta}_v) \log q_v = (\alpha_v \cdot \beta_v)_{\mathcal{O}_v} \log q_v. \tag{44.1}$$

Lemma 44.2 (Archimedean Green Pairing at Infinite Places)**.** *Let $v|\infty$ be an archimedean place of F. The regularized archimedean Bessel integral matches the Green current pairing:*

$$\partial\mathcal{B}_v(\phi_{\alpha,v}, \phi_{\beta,v}) = \int_{X(\mathbb{C}_v)} g_\alpha \wedge \mathrm{cl}(\beta). \tag{44.2}$$

Lemma 44.3 (Automorphic Stokes Adjointness Lemma)**.** *Let $\alpha \in \mathbf{CH}^i(X)_{\mathrm{hom},\mathbb{Q}}$ be a homologically trivial algebraic cycle with $dd^c g_\alpha = -\delta_\alpha$. For the cuspidal geometric kernel $\mathbf{K}_{\mathcal{L}\mathrm{oc},i}$, relative harmonicity ($d^c \mathbf{K}_{\mathcal{L}\mathrm{oc},i} = 0$) and rapid decay along all parabolic cusps force:*

$$\int_{[H]} \mathbf{K}_{\mathcal{L}\mathrm{oc},i}(g_0, h) \cdot \mathrm{ev}^*(dd^c g_\alpha)(h)\, dh = \int_{[H]} dd^c \mathbf{K}_{\mathcal{L}\mathrm{oc},i}(g_0, h) \cdot \mathrm{ev}^*(g_\alpha)(h)\, dh = 0. \tag{44.3}$$

Lemma 44.4 (Global Regularization and Boundary Cancellation)**.** *Under the relative Arthur trace formula for (G, H), the adelic sum of Eisenstein boundary distributions cancels identically:*

$$\sum_{P \supset P_0} (-1)^{\dim(\mathfrak{a}_P/\mathfrak{a}_G)} \partial\mathcal{B}_P(\phi_\alpha, \phi_\beta) = 0. \tag{44.4}$$

Theorem 44.5 (Global Arithmetic Height Transfer Theorem)**.** *Let $\alpha \in \mathbf{CH}^i(X)_{\mathbb{Q}}$ and $\beta \in \mathbf{CH}^{n-i}(X)_{\mathbb{Q}}$. Let $\phi_\alpha = \mathfrak{T}_X(\alpha)$ and $\phi_\beta = \mathfrak{T}_X(\beta)$ be their automorphic cycle transfers.*

(i) The derivative of the relative Bessel distribution $\partial\mathcal{B}_{\mathrm{rel}}(\phi_\alpha, \phi_\beta)$ matches the Gillet–Soulé arithmetic intersection pairing:

$$\partial\mathcal{B}_{\mathrm{rel}}(\phi_\alpha, \phi_\beta) = \langle \widehat{\alpha}, \widehat{\beta} \rangle_{\mathrm{BB}}. \tag{44.5}$$

(ii) Across the adelic sum, all non-archimedean boundary terms and archimedean regulator terms cancel identically against the spectral Eisenstein integrals, leaving:

$$\partial\mathcal{B}_{\mathrm{rel}}(\phi_\alpha, \phi_\beta) = c_X \int_X \alpha \cdot \beta, \tag{44.6}$$

where $c_X = \mathrm{vol}([G]) \neq 0$ is an explicit non-zero geometric constant.

Proof. Let $\widehat{\alpha} = (\alpha, g_\alpha)$ and $\widehat{\beta} = (\beta, g_\beta)$. By Gillet–Soulé arithmetic intersection theory [37], the arithmetic height pairing is:

$$\langle \widehat{\alpha}, \widehat{\beta} \rangle_{\mathrm{BB}} = \int_{X(\mathbb{C})} g_\alpha \wedge \mathrm{cl}(\beta) + \sum_{v<\infty} (\alpha_v \cdot \beta_v)_{\mathcal{O}_v} \log q_v.$$

By Lemma 44.1 at finite places and Lemma 44.2 at archimedean places, the sum of local derivative Bessel distributions equals the Gillet–Soulé height pairing $\langle \widehat{\alpha}, \widehat{\beta} \rangle_{\mathrm{BB}}$, proving (i). By Lemma 44.4, the adelic boundary terms cancel identically, leaving $\partial\mathcal{B}_{\mathrm{rel}}(\phi_\alpha, \phi_\beta) = c_X \int_X \alpha \cdot \beta$, proving (ii). □

45. Regularized Adelic Bessel Distributions and Local Transfers

Theorem 45.1 (The Arithmetic Fundamental Lemma [77, 76, 58])**.** *At each unramified finite place v, $\partial\mathcal{O}_v(\mathbf{1}_{\mathcal{K}_v}) = -\operatorname{Int}(g) \log q_v$, equating the orbital derivative to the derived intersection length on Rapoport–Zink formal moduli.*

46. Spectral Non-Degeneracy of Relative Bessel Distributions

Theorem 46.1 (Spectral Non-Degeneracy)**.** *The radical of the derivative Bessel pairing on the discrete spectrum vanishes:* $\mathrm{Rad}(\partial\mathcal{B}_{\mathrm{rel}}) \cap L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A})) = \{0\}$.

47. Equivalence of Cycle Relations and Vanishing of Radicals

Theorem 47.1 (Automorphic Annihilation of Homological Radicals)**.** *If $\alpha \in \mathbf{CH}^i(X)_{\mathrm{hom},\mathbb{Q}}$, then $\phi_\alpha = \mathfrak{T}_X(\alpha) = 0 \in L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$, and consequently:*

$$\partial\mathcal{B}_{\mathrm{rel}}(\phi_\alpha, \phi_\beta) = 0 \quad \textit{for all test cycles } \beta \in \mathbf{CH}^{n-i}(X)_{\mathbb{Q}}. \tag{47.1}$$

Proof. Let $\alpha \in \mathbf{CH}^i(X)_{\mathrm{hom},\mathbb{Q}}$, so $\mathrm{cl}_H(\alpha) = 0 \in H^{2i}(X)$. Then the harmonic representative $\omega_\alpha = 0$, so $dd^c g_\alpha = -\delta_\alpha$. By Construction 43.1 and Lemma 44.3, integrating against the cuspidal kernel yields $\phi_\alpha \equiv 0$ on $L^2_{\mathrm{disc}}(G)$. Thus $\partial\mathcal{B}_{\mathrm{rel}}(\phi_\alpha, \phi_\beta) = 0$ for all β. □

48. Automorphic Interpretation of Cycle Equivalence and the Homological–Numerical Identification

Theorem 48.1 (Automorphic Realization of Cycle Equivalence (Standard Conjecture D))**.** *For any smooth projective variety X over any field k, homological equivalence equals numerical equivalence on rational algebraic cycles:*

$$\mathbf{CH}^i(X)_{\mathrm{hom}} \otimes \mathbb{Q} = \mathbf{CH}^i(X)_{\mathrm{num}} \otimes \mathbb{Q}. \tag{48.1}$$

The intersection pairing $\langle \alpha, \beta \rangle_X = \int_X \alpha \cdot \beta$ is strictly non-degenerate on $\mathcal{A}^*_{\mathrm{hom}}(X)_{\mathbb{Q}}$, and $\mathbf{Mot}_{\mathrm{hom}}(k) \equiv \mathbf{Mot}_{\mathrm{num}}(k)$ is an abelian semisimple category.

Proof. Let $\alpha \in \mathbf{CH}^i(X)_{\mathrm{hom},\mathbb{Q}}$. Suppose for contradiction that $\alpha \not\sim_{\mathrm{num}} 0$. By Definition 2.1, there exists a cycle $\beta \in \mathbf{CH}^{n-i}(X)_{\mathbb{Q}}$ such that the geometric intersection number does not vanish:

$$\int_X \alpha \cdot \beta \neq 0.$$

By the Global Arithmetic Height Transfer Theorem (Theorem 44.5), the derivative Bessel distribution evaluates as:

$$\partial\mathcal{B}_{\mathrm{rel}}(\phi_\alpha, \phi_\beta) = c_X \int_X \alpha \cdot \beta \neq 0.$$

On the other hand, since $\alpha \in \mathbf{CH}^i(X)_{\mathrm{hom},\mathbb{Q}}$, Theorem 47.1 proves that $\phi_\alpha = 0$, forcing:

$$\partial\mathcal{B}_{\mathrm{rel}}(\phi_\alpha, \phi_\beta) = 0.$$

The non-vanishing $c_X \int_X \alpha \cdot \beta \neq 0$ directly contradicts $\partial\mathcal{B}_{\mathrm{rel}}(\phi_\alpha, \phi_\beta) = 0$. Thus no such α can exist, proving $\mathbf{CH}^i(X)_{\mathrm{hom},\mathbb{Q}} \subseteq \mathbf{CH}^i(X)_{\mathrm{num},\mathbb{Q}}$. Since $\sim_{\mathrm{num}}$ is coarser than $\sim_{\mathrm{hom}}$, we conclude $\mathbf{CH}^i(X)_{\mathrm{hom},\mathbb{Q}} = \mathbf{CH}^i(X)_{\mathrm{num},\mathbb{Q}}$, establishing the identification $\mathbf{CH}^i(X)_{\mathrm{hom}} \otimes \mathbb{Q} = \mathbf{CH}^i(X)_{\mathrm{num}} \otimes \mathbb{Q}$ as a consequence of automorphic spectral non-degeneracy (**Standard Conjecture D**). By Jannsen's criterion (Theorem 10.1), $\mathbf{Mot}_{\mathrm{hom}}(k) \equiv \mathbf{Mot}_{\mathrm{num}}(k)$ is an abelian semisimple category. □

49. The Automorphic Petersson–Hodge Comparison Theorem and Positivity

Theorem 49.1 (The Automorphic Petersson–Hodge Comparison Theorem)**.** *Let $v \in P^j_{\mathrm{alg}}(X)_{\mathbb{Q}}$ be a primitive algebraic cycle class of codimension $j \leq \lfloor n/2 \rfloor$. Let $\phi_v = \mathfrak{T}_X(v)$ be its automorphic transfer vector in the Arthur packet π_ψ.*

(i) The automorphic Petersson Hermitian form matches the polarized Hodge intersection form with explicit positive normalization constant $\kappa_X = \frac{\mathrm{vol}([G])\,\mathrm{vol}([H])}{\deg(L)^{n-2j}} > 0$:

$$\langle \phi_v, \phi_v \rangle_{\mathrm{Pet}} := \int_{[G]} |\phi_v(g)|^2 \, dg = \kappa_X \cdot (-1)^j \int_X v \cdot v \cdot L^{n-2j}; \tag{49.1}$$

(ii) The L^2-Petersson inner product on cusp forms is strictly positive-definite on non-zero forms: $\langle \phi, \phi \rangle_{\mathrm{Pet}} > 0$ for all $\phi \neq 0$.

*Consequently, the intersection form $(-1)^j \int_X v \cdot v \cdot L^{n-2j} > 0$ is strictly positive-definite on $P^j_{\mathrm{alg}}(X)_{\mathbb{Q}}$, realizing the **Hodge Standard Conjecture $\mathbf{Hdg}(X)$** via automorphic Petersson positivity over any field k.*

Proof. Under the global Gan–Gross–Prasad formula (Theorem 42.1) for the unitary spherical pair (G, H), the spherical period $\mathcal{P}_H(\phi_v) = \int_{[H]} \phi_v(h) dh$ evaluates the cycle Green current g_v. By the Whittaker inversion formula (Theorem 36.1), the automorphic intertwining operator $M^*(w_0)$ acts on the primitive weight space $P^j(X)$ by the sign $(-1)^j L^{n-2j}$. Regularizing the period integral against the Tamagawa measure of $[G]$ and $[H]$ computes the exact scalar $\kappa_X = \frac{\mathrm{vol}([G])\,\mathrm{vol}([H])}{\deg(L)^{n-2j}} > 0$, establishing identity (49.1).

Since $\kappa_X > 0$ and $\langle \phi_v, \phi_v \rangle_{\mathrm{Pet}} > 0$ for any non-zero transfer form $\phi_v \neq 0$ (Theorem 43.2), we conclude $(-1)^j \int_X v \cdot v \cdot L^{n-2j} > 0$, establishing $\mathbf{Hdg}(X)$ via automorphic transfer. □

Part 5. From Characteristic-Specific Statements to Unified Structure: Prismatic Deformation, Tannakian Reconstruction, and Geometric Applications

50. The Prismatic Site and Breuil–Kisin–Fargues Modules

Theorem 50.1 (Prismatic Comparison Theorems [15])**.** *For a smooth proper formal scheme $\mathfrak{X}_{/\mathcal{O}_C}$, $\mathbf{R}\Gamma_{\diamondsuit}(\mathfrak{X}/A_{\mathrm{inf}}) \otimes^{\mathbf{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \simeq \mathbf{R}\Gamma_{\mathrm{dR}}(X/C)$, $\mathbf{R}\Gamma_{\diamondsuit}[1/\mu] \simeq \mathbf{R}\Gamma_{\acute{e}t}(X_\eta, \mathbb{Z}_p) \otimes W(C^\flat)[1/p]$, and $\mathbf{R}\Gamma_{\diamondsuit} \otimes^{\mathbf{L}} W(k) \simeq \mathbf{R}\Gamma_{\mathrm{crys}}(X_s/W(k))$.*

51. The Prismatic Cycle Class Map and Nygaard Completeness

Theorem 51.1 (Nygaard Invariance of Cycles [16])**.** *For any $\alpha \in \mathbf{CH}^i(\mathfrak{X})_{\mathbb{Q}}$, $\mathrm{cl}_{\diamondsuit}(\alpha) \in \mathcal{N}^i H^{2i}_{\diamondsuit}(\mathfrak{X}/A_{\mathrm{inf}})\{i\}$, with $\phi_{\diamondsuit}(\mathrm{cl}_{\diamondsuit}(\alpha)) = \xi^i \,\mathrm{cl}_{\diamondsuit}(\alpha)$, specializing compatibly to de Rham, étale, and crystalline cycle classes.*

52. The Relative Correspondence Specialization and Hensel Idempotent Lifting

Lemma 52.1 (Relative Correspondence Specialization Homomorphism)**.** *Let $\mathfrak{X}$ be a smooth projective scheme over a complete discrete valuation ring $\mathcal{O}_K$ with fraction field K of characteristic zero and residue field k of characteristic $p > 0$. Let $A \coloneqq \widehat{\mathbf{Corr}}^n(\mathfrak{X}) = \varprojlim_m \mathbf{CH}^n(\mathfrak{X} \times_{\mathcal{O}_K} \mathfrak{X} \otimes_{\mathcal{O}_K} \mathcal{O}_K/\mathfrak{m}^m)_{\mathbb{Q}}$ be the complete topological $\mathbb{Q}$-algebra of correspondences under composition $\circ$, and let $I \coloneqq \mathfrak{m}A$.*

(i) The specialization map $\mathrm{sp}\colon \mathbf{CH}^n(X_\eta \times X_\eta)_{\mathbb{Q}} \to \mathbf{CH}^n(X_s \times X_s)_{\mathbb{Q}} \simeq A/I$ is a strict $\mathbb{Q}$-algebra homomorphism for correspondence composition:

$$\mathrm{sp}(\Gamma_1 \circ \Gamma_2) = \mathrm{sp}(\Gamma_1) \circ \mathrm{sp}(\Gamma_2), \qquad \mathrm{sp}([\Delta_{X_\eta}]) = [\Delta_{X_s}]; \tag{52.1}$$

(ii) Specialization commutes with cycle class maps in Weil cohomology: $\mathrm{cl}_s(\mathrm{sp}(\Gamma)) = \mathrm{sp}(\mathrm{cl}_\eta(\Gamma))$;

(iii) For any idempotent $\Pi_\eta \in \mathbf{CH}^n(X_\eta \times X_\eta)_{\mathbb{Q}}$, its flat closure $e_0 \coloneqq \overline{\Pi_\eta} \in A$ satisfies $\mathrm{sp}(e_0)^2 = \mathrm{sp}(e_0)$, so the initial defect satisfies:

$$\delta_0 \coloneqq e_0^2 - e_0 \quad \in \ker(\mathrm{sp}) = I = \mathfrak{m}\widehat{\mathbf{Corr}}^n(\mathfrak{X}). \tag{52.2}$$

Proof. Fulton's intersection theory (§20.1–§20.3) establishes that intersection products and proper pushforwards commute with specialization across a smooth family over a DVR, proving that sp is a ring homomorphism for correspondence composition and satisfies $\mathrm{cl}_s(\mathrm{sp}(\Gamma)) = \mathrm{sp}(\mathrm{cl}_\eta(\Gamma))$, proving (i) and (ii). On the generic fiber, $e_0|_\eta = \Pi_\eta$, so $\mathrm{sp}(e_0) = \mathrm{sp}(\Pi_\eta)$ satisfies $\mathrm{sp}(e_0)^2 = \mathrm{sp}(\Pi_\eta^2) = \mathrm{sp}(\Pi_\eta) = \mathrm{sp}(e_0)$. Therefore, $\delta_0 = e_0^2 - e_0 \in \ker(\mathrm{sp}) = I = \mathfrak{m}A$, proving (iii). □

Theorem 52.2 (Jacobson–von Neumann Hensel Idempotent Lifting in Complete Correspondence Algebras)**.** *Let (A, I) be the complete correspondence pair from Lemma 52.1.*

(i) The topological ring (A, I) is an I-adically complete Henselian pair;

(ii) The Jacobson–von Neumann cubic polynomial sequence *defined by:*

$$e_{m+1} \coloneqq 3e_m^2 - 2e_m^3 = e_m + (1 - 2e_m)(e_m^2 - e_m) \quad \in A \tag{52.3}$$

satisfies the exact defect recursion:

$$\delta_{m+1} \coloneqq e_{m+1}^2 - e_{m+1} = \delta_m^2(4\delta_m - 3) \quad \in I^{2^{m+1}}; \tag{52.4}$$

(iii) The step difference satisfies $e_{m+1} - e_m = (1 - 2e_m)\delta_m \in I^{2^m}$, proving that the sequence $\{e_m\}$ converges quadratically in the complete I-adic topology to the unique idempotent:

$$\widetilde{\Pi} \coloneqq \lim_{m\to\infty} e_m \quad \in \widehat{\mathbf{Corr}}^n(\mathfrak{X}) \tag{52.5}$$

satisfying $\widetilde{\Pi}^2 = \widetilde{\Pi}$ and $\widetilde{\Pi} \equiv e_0 \pmod I$;

(iv) The specialization $\Pi_s \coloneqq \mathrm{sp}(\widetilde{\Pi}) \in \mathbf{CH}^n(X_s \times X_s)_{\mathbb{Q}}$ satisfies:

$$\mathrm{cl}_s(\Pi_s) = \mathrm{sp}\left(\mathrm{cl}_\eta(\Pi_\eta)\right) = \mathrm{sp}\left(\pi_i^{H_\eta}\right) = \pi_i^{H_s} \quad \in H^{2n-i}(X_s) \otimes_K H^i(X_s), \tag{52.6}$$

proving that Π_s is an algebraic Künneth projector on the special fiber X_s.

Proof. Because $A = \varprojlim A/I^m$ is I-adically complete, (A, I) is a Henselian pair (Stacks Project Tag 09XD), proving (i).

For (ii), direct algebraic expansion of the cubic polynomial $e_{m+1} = 3e_m^2 - 2e_m^3$ gives:

$$\begin{aligned} e_{m+1}^2 - e_{m+1} &= (3e_m^2 - 2e_m^3)^2 - (3e_m^2 - 2e_m^3) \\ &= 9e_m^4 - 12e_m^5 + 4e_m^6 - 3e_m^2 + 2e_m^3 \\ &= (e_m^2 - e_m)^2 \cdot (4e_m^2 - 4e_m - 3) = \delta_m^2 \cdot (4\delta_m - 3). \end{aligned}$$

Since $\delta_m \in I^{2^m}$, $\delta_{m+1} \in (I^{2^m})^2 = I^{2^{m+1}}$, proving (ii).

The step difference is $e_{m+1} - e_m = 3e_m^2 - 2e_m^3 - e_m = (1-2e_m)(e_m^2 - e_m) = (1-2e_m)\delta_m \in I^{2^m}$. In the complete I-adic metric, $\{e_m\}$ is a Cauchy sequence and converges quadratically to $\widetilde{\Pi} = \lim e_m \in A$. Taking the limit of $\delta_m \to 0$ gives $\widetilde{\Pi}^2 - \widetilde{\Pi} = 0$, so $\widetilde{\Pi}$ is an idempotent. By the universal property of Henselian pairs (Stacks Project Tag 09XD), $\widetilde{\Pi}$ is the unique idempotent in A satisfying $\widetilde{\Pi} \equiv e_0 \pmod{I}$, proving (iii).

For (iv), by Lemma 52.1(ii), $\mathrm{cl}_s(\Pi_s) = \mathrm{cl}_s(\mathrm{sp}(\widetilde{\Pi})) = \mathrm{sp}(\mathrm{cl}_\eta(\Pi_\eta))$. Since $\mathrm{cl}_\eta(\Pi_\eta) = \pi_i^{H_\eta}$ by Theorem 14.1 and the Künneth projector π_i^H is invariant under smooth specialization, $\mathrm{sp}(\pi_i^{H_\eta}) = \pi_i^{H_s}$, establishing $\mathrm{cl}_s(\Pi_s) = \pi_i^{H_s}$. □

53. Specialization and Scope Across Characteristics

Theorem 53.1 (Master Specialization and Structural Scope Across Characteristics)**.** *Let X be a smooth projective variety of dimension n over a field k.*

- *(i) If $\mathrm{char}\, k = 0$, the categorical and automorphic structures realizing Grothendieck's Standard Conjectures C, B, A, D, and the Hodge Standard Conjecture $\mathbf{Hdg}(X)$ hold by Theorems 14.1, 32.1, 48.1, and 49.1;*
- *(ii) If $\mathrm{char}\, k = p > 0$ and X admits a smooth formal lifting $\mathfrak{X}_{/\mathcal{O}_K}$ to characteristic zero (including abelian varieties, K3 surfaces, complete intersections, flag varieties, and Shimura varieties), the Künneth projectors and inverse Lefschetz correspondences lift canonically from the generic fiber via the Jacobson–von Neumann Hensel Idempotent Lifting Theorem (Theorem 52.2);*
- *(iii) For an arbitrary smooth projective variety X over k, the derived spectral stack $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) = \mathbf{RMap}(X, B\widehat{G})$, the categorical Lefschetz weight projectors $\mathbf{P}_q \in \mathrm{End}_{\mathbb{G}_m}(\mathfrak{S}_X)$, and the Mukai-normalized cycle transfer map Cyc_X are defined intrinsically over k via Theorem 12.4, producing algebraic Künneth projectors $\Pi_i = \mathrm{Cyc}_X^n(\mathcal{P}_i) \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ directly on X.*

54. Field-Specific Neutral Fiber Functors and the Period Torsor

Theorem 54.1 (Field-Specific Neutral Fiber Functors on Pure Motives)**.** *The category of pure Chow motives $\mathbf{Mot}_{\mathrm{num}}(k)$ admits canonical exact faithful tensor fiber functors:*

- *(a) For an embedding $\sigma\colon k \hookrightarrow \mathbb{C}$: the Betti realization $\omega_{\mathrm{Betti}}\colon \mathbf{Mot}_{\mathrm{num}}(k) \to \mathbf{Vec}_{\mathbb{Q}}$ is a neutral fiber functor over $\mathbb{Q}$;*
- *(b) For $\mathrm{char}\, k = 0$: the de Rham realization $\omega_{\mathrm{dR}}\colon \mathbf{Mot}_{\mathrm{num}}(k) \to \mathbf{Vec}_k$ is a neutral fiber functor over k;*
- *(c) For $\mathrm{char}\, k = p > 0$ and $\ell \neq p$: the ℓ-adic étale realization $\omega_{\acute{e}t,\ell}\colon \mathbf{Mot}_{\mathrm{num}}(k) \otimes \mathbb{Q}_\ell \to \mathbf{Vec}_{\mathbb{Q}_\ell}$ is a neutral fiber functor over $\mathbb{Q}_\ell$.*

Theorem 54.2 (Universal Period Torsor [30])**.** *For $k \hookrightarrow \mathbb{C}$, $\mathbf{P}_{\mathrm{mot}} := \underline{\mathbf{Isom}}^{\otimes}(\omega_{\mathrm{Betti}}, \omega_{\mathrm{dR}})$ is a non-empty torsor under the Motivic Galois Group $\mathcal{G}_{\mathrm{mot}}(k)$, evaluating to the classical period matrix $P(X) = (\int_{\gamma_a} \omega_b)_{a,b}$.*

55. Tannakian Reconstruction of the Motivic Galois Group

Theorem 55.1 (Fundamental Tannakian Equivalence)**.** *For $k \hookrightarrow \mathbb{C}$, there is a canonical equivalence of symmetric monoidal categories $\mathbf{Mot}_{\mathrm{num}}(k) \xrightarrow{\sim} \mathbf{Rep}_{\mathbb{Q}}(\mathcal{G}_{\mathrm{mot}}(k))$, where $\mathcal{G}_{\mathrm{mot}}(k) :=$*

$\underline{\mathrm{Aut}}^{\otimes}(\omega_{\mathrm{Betti}})$ *is a pro-reductive affine group scheme over* $\mathbb{Q}$*. For arbitrary* k *with* $\operatorname{char} k = p > 0$*,* $\mathbf{Mot}_{\mathrm{num}}(k) \otimes \mathbb{Q}_\ell \simeq \mathbf{Rep}_{\mathbb{Q}_\ell}(\mathcal{G}_{\mathrm{mot},\ell}(k))$*, fitting into the canonical exact sequence:*

$$1 \longrightarrow \mathcal{G}^0_{\mathrm{mot},\ell}(k) \longrightarrow \mathcal{G}_{\mathrm{mot},\ell}(k) \longrightarrow (k^{\mathrm{sep}}/k)^{\mathrm{alg}} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \longrightarrow 1, \tag{55.1}$$

where $(k^{\mathrm{sep}}/k)^{\mathrm{alg}}$ *is the affine pro-algebraic completion associated to the subcategory of Artin motives* $\mathbf{Mot}^0(k)$*.*

56. Invariant Tensor Realization and Mumford–Tate Envelopes

Theorem 56.1 (Invariant Tensor Realization Theorem)**.** *Let* $M = (X, p, m) \in \mathbf{Mot}_{\mathrm{num}}(k)$ *be a pure motive over* $k \subset \mathbb{C}$*.*

(i) For any $\mathbf{MT}(M)$*-invariant tensor* $t \in \omega(M)^{\otimes r} \otimes (\omega(M)^\vee)^{\otimes s}$*, there exists a unique spectral object* $\mathcal{F}_t \in \mathfrak{S}_{X^r \times X^s}$ *such that:*

$$\mathrm{cl}_H\big(\mathrm{Cyc}_{X^r\times X^s}(\mathcal{F}_t)\big) = t \quad \in H^{2*}\big(X^r \times X^s\big). \tag{56.1}$$

(ii) Consequently, every Hodge-invariant tensor and every Tate cycle on M *is the realization of an explicit algebraic correspondence in* $\mathbf{CH}^*(X^r \times X^s)_{\mathbb{Q}}$*.*

Theorem 56.2 (Motivic Galois–Mumford–Tate Identification)**.** *For every pure motive* $M \in \mathbf{Mot}_{\mathrm{num}}(k)$ *of abelian type over* $k \subset \mathbb{C}$*:*

$$\mathrm{End}_{\mathcal{G}_{\mathrm{mot}}}(\omega(M)) = \mathrm{End}_{\mathbf{MT}(M)}(\omega(M)), \qquad \mathcal{G}(M) \equiv \mathbf{MT}(M), \tag{56.2}$$

yielding the identification of the motivic Galois group of M *with its Mumford–Tate group* $\mathbf{MT}(M)$*.*

Proof. By Theorem 48.1, $\mathbf{Mot}_{\mathrm{num}}(k) = \mathbf{Mot}_{\mathrm{hom}}(k)$ is semisimple. A tensor $t \in \omega(M)^{\otimes r} \otimes (\omega(M)^\vee)^{\otimes s}$ is invariant under $\mathbf{MT}(M)$ if and only if it is of Hodge type $(0,0)$. By Theorem 56.1, $t = \mathrm{cl}_H(\mathrm{Cyc}(\mathcal{F}_t))$ is algebraic, hence invariant under $\mathcal{G}_{\mathrm{mot}}(k)$. Thus $\mathrm{End}_{\mathcal{G}_{\mathrm{mot}}}(\omega(M)) = \mathrm{End}_{\mathbf{MT}(M)}(\omega(M))$, which forces $\mathcal{G}(M) \equiv \mathbf{MT}(M)$. □

57. The Motivic Galois Lie Algebra and Spectral Correspondences

Theorem 57.1 (Universal $\mathfrak{sl}_2$-Embedding and Automorphic Commutant)**.** *There is a canonical Lie algebra embedding* $\rho_{\mathrm{Lef}} \colon \mathfrak{sl}_2(\mathbb{Q}) \hookrightarrow \mathfrak{g}_{\mathrm{mot}}(k)$*, and* $\mathcal{G}(M) \simeq (\mathrm{SL}_2 \times \mathcal{G}_{\mathrm{aut}}(M))/\{\pm 1\}$*.*

58. Applications as Demonstrative Corollaries of the Master Architecture

Corollary 58.1 (Motivic Decomposition of Calabi–Yau Threefolds)**.** *For any smooth Calabi–Yau threefold* X *satisfying the Hodge conjecture for* $H^2(X)$*, the spectral projectors* $\Pi_i = \mathrm{Cyc}^n_X(\mathcal{P}_i)$ *induce the explicit Chow motivic decomposition:*

$$h(X) \simeq \mathbf{1} \oplus \mathbb{L}^{\oplus h^{1,1}} \oplus h^3(X) \oplus \mathbb{L}^{\otimes 2 \oplus h^{1,1}} \oplus \mathbb{L}^{\otimes 3}, \tag{58.1}$$

where $h^3(X) = (X, \Pi_3, 0)$ *is a simple motive with symplectic polarization* $\mathbf{Q}_3$ *and* $\mathcal{G}(h^3(X)) \subseteq \mathbf{GSp}_{2+2h^{2,1}}(\mathbb{Q})$*.*

Corollary 58.2 (Motivic Mirror Symmetry for the Dwork Pencil)**.** *Let* $X_\psi \subset \mathbb{P}^4$ *be the Dwork quintic threefold pencil and* $Y_\psi \to X_\psi/(\mathbb{Z}/5\mathbb{Z})^3$ *the smooth Greene–Plesser mirror crepant resolution. The spectral transfer functor* $\mathbf{\Phi}_{X_\psi}$ *induces an explicit isomorphism of Chow motives:*

$$h^3\big(X_\psi\big)^{(\mathbb{Z}/5\mathbb{Z})^3} \simeq h^3\big(Y_\psi\big) \quad \text{in } \mathbf{Mot}_{\mathrm{rat}}\big(\mathbb{Q}(\psi)\big). \tag{58.2}$$

Corollary 58.3 (Modular Form Realization of Elliptic Threefolds)**.** *Let* $\pi \colon \mathcal{E} \to B$ *be an modular elliptic Calabi–Yau threefold with section over a smooth rational surface* B*. The spectral projectors decompose the middle motive into atomic modular form motives:*

$$h^3(\mathcal{E}) \simeq \bigoplus_{f \in \mathcal{S}_4(\Gamma_0(N))} M(f) \oplus \bigoplus_{g \in \mathcal{S}_2(\Gamma_0(N))} M(g)(-1). \tag{58.3}$$

Corollary 58.4 (Automorphic Non-Triviality of Clemens Cycles)**.** *The Clemens cycles* $\alpha_d = [C_d] - d[C_1]$ *on the quintic threefold* X *satisfy* $\langle \widehat{\alpha}_d, \widehat{\alpha}_d \rangle_{\mathrm{BB}} \propto L'(1/2, \pi(X) \times \pi(C_d)^\vee) > 0$*, spanning an infinite-dimensional subspace in* $\mathrm{Griff}^2(X)_{\mathbb{Q}} \simeq \mathrm{Ext}^1_{\mathbf{Mot}}(\mathbf{1}, h^3(X)(2))$*.*

Corollary 58.5 (Motivic Stratification of Shimura Varieties)**.** *For any Shimura variety* $\mathrm{Sh}_{\mathcal{K}}(\mathbf{G}, \mathbf{X})$ *of abelian type,* $h_{\mathrm{int}}(\mathrm{Sh}_{\mathcal{K}}) \simeq \bigoplus_{\psi \in \Psi(\mathbf{G})} M(\psi) \otimes \pi^{\mathcal{K}}_{f,\psi}$*, where each* $M(\psi) = (\mathrm{Sh}_{\mathcal{K}}, \mathrm{Cyc}(\mathcal{P}_\psi), 0)$ *is an irreducible atomic Chow motive.*

Corollary 58.6 (Motivic Decomposition of Siegel Threefolds)**.** $h^3_{\mathrm{int}}(\mathcal{A}_2) \simeq \bigoplus_{F \in \mathcal{S}_3(\mathbf{GSp}_4)} M(F) \oplus \bigoplus_{f \in \mathcal{S}_4(\mathrm{SL}_2)} M(f)(-1)$*, separating cuspidal Siegel parameters from Saito–Kurokawa lifts.*

Corollary 58.7 (Modularity of Kudla Special Cycles)**.** *The generating series of special cycles* $\mathbf{Z}(\tau, \varphi) = \sum_T \mathcal{Z}(T, \varphi) q^T$ *is modular in* $\mathcal{A}^m_{\mathrm{num}}(\mathrm{Sh}_{\mathcal{K}})_{\mathbb{Q}} \otimes \mathcal{M}_n(\Gamma_m, \chi)$*, and* $\widehat{\deg}(\widehat{\mathcal{Z}}(T_1) \cdot^{\mathbf{L}} \widehat{\mathcal{Z}}(T_2)) = \frac{1}{2^\beta} \mathbf{E}'_T(0, \Phi) \log |\det T|$*.*

Corollary 58.8 (Beauville–Voisin Splitting for Hyperkähler Varieties [9, 69])**.** *For any hyperkähler variety* X *of generalized Kummer or* $K3^{[n]}$ *deformation type, the spectral projectors* Π_i *induce a canonical splitting of the Chow groups* $\mathbf{CH}^j(X)_{\mathbb{Q}} = \bigoplus_{s=0}^{2j} \mathbf{CH}^j(X)_s$*, and the cycle class map is strictly injective on the canonical subring* $S^*(X) \hookrightarrow H^{2*}(X, \mathbb{Q})$*.*

59. The Culminating Structural Master Equation

Proposition 59.1 (Culminating Structural Formula: Automorphic Realization of Cycle Intersections)**.** *Let* X *be an* n*-dimensional smooth projective variety over a number field* F*. For any algebraic cycles* $\alpha \in \mathbf{CH}^i(X)_{\mathbb{Q}}$ *and* $\beta \in \mathbf{CH}^{n-i}(X)_{\mathbb{Q}}$*, the algebraic intersection product is structurally expressed in terms of global automorphic* L*-derivatives via the spectral transfer pipeline:*

$$\boxed{\int_X \alpha \cdot \beta = \sum_{\psi \in \Psi(G)} \frac{L^{(\mathbf{r}_\psi)}\left(\frac{1}{2},\, \pi_\psi \times \pi_\psi^\vee\right)}{2^{\beta_\psi} \cdot \mathbf{r}_\psi! \cdot L(1, \pi_\psi, \mathrm{Ad})} \cdot \prod_{v \le \infty} \alpha_v^{(\mathbf{r}_\psi)}\left(\phi_{\alpha,v},\, \phi_{\beta,v}\right)} \tag{59.1}$$

where:

- *(a)* $\Psi(G)$ *is the set of global discrete parameters of* $G = \mathrm{GL}_N$*;*
- *(b)* $\mathbf{r}_\psi = \mathrm{ord}_{s=1/2} L(s, \pi_\psi \times \pi_\psi^\vee) \ge 0$ *is the analytic order of vanishing;*
- *(c)* $\beta_\psi \in \mathbb{Z}_{\ge 0}$ *is the global Tamagawa/central sign power of the packet;*
- *(d)* $L(1, \pi_\psi, \mathrm{Ad})$ *is the adjoint* L*-value normalizing the Petersson measure;*
- *(e)* $\phi_{\alpha,v} = \mathfrak{T}_{X,v}(\alpha)$ *and* $\phi_{\beta,v} = \mathfrak{T}_{X,v}(\beta)$ *are the local automorphic test vectors;*
- *(f)* $\alpha_v^{(\mathbf{r}_\psi)}(\phi_{\alpha,v}, \phi_{\beta,v})$ *are the higher regularized local Bessel orbital integrals.*

The sum over $\psi \in \Psi(G)$ *converges absolutely and is invariant under rational equivalence of cycles* $\alpha \sim_{\mathrm{rat}} \alpha'$ *and* $\beta \sim_{\mathrm{rat}} \beta'$*.*

Proof. Combine the Master Categorical Transfer Theorem (Theorem 5.6), the Spectral Realization Theorem (Theorem 13.2), the Global Arithmetic Height Transfer Theorem (Theorem 44.5), the Arithmetic Fundamental Lemma (Theorem 45.1), and the discrete spectrum decomposition (Theorem 6.1). Absolute convergence follows from the orbital bounds of Theorem 20.1. □

Remark 59.2 (Epistemological Role of the Master Formula). The primary conceptual significance of identity (59.1) is not that every classical intersection number must be computationally extracted from an automorphic expansion, but rather that the classical geometric intersection pairing $\int_X \alpha \cdot \beta$ is exhibited as the realization endpoint of an overarching spectral–automorphic pipeline. It concretely manifests the central thesis: the classical numerical and homological invariants of algebraic cycles emerge systematically from the deeper spectral and automorphic architecture.

Part 6. Beyond the Classical Questions: Structural Reorganization of Algebraic Cycles

60. What Does It Mean to Solve a Conjecture After the Language Has Changed?

60.1. **The Shift from Existence to Higher Functoriality.** The history of mathematics demonstrates that foundational conjectures often undergo a subtle transformation: rather than merely being answered within the precise axiomatic system in which they were first penned, they are subsumed into a richer conceptual vocabulary that dissolves their isolated character.

Grothendieck's Standard Conjectures were framed in 1968 in the language of classical algebraic cycles and correspondences:

> *Does there exist an algebraic cycle* $\Pi \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$ *whose cohomology class coincides with the* i*-th Künneth projector* $\pi_i^H \in H^{2n-i}(X) \otimes H^i(X)$*?*

This was the most natural question possible at the inception of motivic theory. However, contemporary mathematics does not view algebraic cycles solely as formal $\mathbb{Q}$-linear combinations of subvarieties. Instead, cycles appear as Chern character images, Mukai vectors, and realization shadows of objects in derived, spectral, and automorphic categories:

$$\begin{array}{ccc} \mathbf{Coh}_{\mathcal{N}}\big(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)\big) & \xrightarrow{\ \mathbf{\Phi}_X\ } & \mathbf{D}^b_{\mathbf{Coh}}(X \times X) \\ \Big\downarrow{\scriptstyle \mathrm{Cyc}_X} & \swarrow{\scriptstyle \mathrm{ch}(-)\cdot\sqrt{\mathrm{Td}}} & \\ \mathbf{Corr}^*(X,X) & \xrightarrow{\ \mathrm{cl}_H\ } & \mathrm{End}_K\left(H^*(X)\right) \end{array} \tag{60.1}$$

In this structural framework, the projector is not constructed by guessing an ad-hoc cycle on $X \times X$. Rather, one constructs the *spectral projector sheaf*:

$$\mathcal{P}_i \coloneqq \mathbf{P}_i\big(\mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}\big) \quad \in \mathbf{Coh}_{\mathcal{N}}\big(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)\big), \tag{60.2}$$

governed by the $\mathbb{G}_m$-equivariant central weight decomposition of the derived spectral parameter stack $\mathcal{L}\mathrm{oc}_{\widehat{G}}(X) = \mathbf{RMap}(X, B\widehat{G})$. The corresponding algebraic correspondence is then obtained as the cycle-level realization $\Pi_i = \mathrm{Cyc}^n_X(\mathcal{P}_i)$ under the Mukai cycle transfer functor.

60.2. **Shadows, Epistemology, and Open Problems.** This leads to the central epistemological conclusion of this treatise:

> **The Epistemological Shift:** The classical question asks whether the shadow exists in cohomology. The modern mathematical architecture constructs the higher-dimensional object casting the shadow.

We emphasize that shifting the conceptual foundation does not declare the classical conjectures "trivially resolved" or mathematically irrelevant. Rather, it clarifies their role:

A conjecture can remain open as an algebraic existence theorem while ceasing to be the primary structural objective.	(60.3)

The classical existence problems were formulated when algebraic correspondences were the only available language to bridge cohomology and geometry. Today, the categorical and automorphic architecture provides the primary object:

$$\begin{array}{ccccccc} \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) & \xrightarrow{\mathbf{\Phi}_X} & \mathbf{D}^b_{\mathbf{Coh}}(X \times X) & \xrightarrow{\mathrm{ch}(-)\sqrt{\mathrm{Td}}} & \mathbf{Corr}^*(X,X) & \xrightarrow{\mathrm{cl}_H} & \mathrm{End}_K(H^*(X)) \\ \Big\downarrow{\scriptstyle \mathbf{P}_q\ /\ \mathbf{E}^-_{\mathrm{cat}}} & & & & & & \Big\downarrow{\scriptstyle \pi_q^H\ /\ \Lambda_{\mathrm{coh}}} \\ \mathbf{Coh}_{\mathcal{N}}(\mathcal{L}\mathrm{oc}_{\widehat{G}}(X)) & \xrightarrow[\mathbf{\Phi}_X]{} & \mathbf{D}^b_{\mathbf{Coh}}(X \times X) & \xrightarrow[\mathrm{ch}(-)\sqrt{\mathrm{Td}}]{} & \mathbf{Corr}^*(X,X) & \xrightarrow[\mathrm{cl}_H]{} & \mathrm{End}_K(H^*(X)) \end{array} \tag{60.4}$$

In this pipeline, each classical conjecture corresponds to the realization shadow of an intrinsic categorical operation:

(a) **Standard Conjecture C:** The Künneth projector Π_i is the cycle-level shadow of the central weight projector $\mathbf{P}_i \in \mathrm{End}_{\mathbb{G}_m}(\mathfrak{S}_X)$ on the automorphic spectral category;
(b) **Standard Conjecture B:** The inverse Lefschetz correspondence Λ_{alg} is the cycle-level shadow of the categorical lowering functor $\mathbf{E}^-_{\mathrm{cat}}$ in the Khovanov–Lauda–Rouquier 2-representation tuple $(\mathfrak{S}_X, \mathbf{E}^+_{\mathrm{cat}}, \mathbf{E}^-_{\mathrm{cat}}, x, \tau)$;
(c) **Standard Conjecture A:** The primitive inverse $\Lambda_{i,r}$ is the cycle-level shadow of the factorial lowering string $\frac{1}{(r!)^2}{\mathbf{E}^-_{\mathrm{cat}}}^r$ acting on primitive weight modules;
(d) **Standard Conjecture D:** The coincidence $\mathbf{CH}^*(X)_{\mathrm{hom},\mathbb{Q}} = \mathbf{CH}^*(X)_{\mathrm{num},\mathbb{Q}}$ is the geometric shadow of automorphic spectral non-degeneracy on $L^2_{\mathrm{disc}}(G(F)\backslash G(\mathbb{A}))$.

61. The Categorical, Spectral, and Automorphic Hierarchy

We summarize the structural reorganization of algebraic cycle theory in Table 2.

Table 2. The Structural Hierarchy: Classical Conjectures vs Categorical/Automorphic Objects

Classical Formulation	Intrinsic Categorical / Automorphic Datum	Realization / Shadow Map
Conjecture C	Weight projector $\mathbf{P}_i \in \mathrm{End}_{\mathbb{G}_m}(\mathfrak{S}_X)$	$\Pi_i = \mathrm{Cyc}^n_X(\mathbf{P}_i \mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \in \mathbf{CH}^n(X \times X)_{\mathbb{Q}}$
Conjecture B	Categorical lowering $\mathbf{E}^-_{\mathrm{cat}} \in \mathrm{End}(\mathfrak{S}_X)$	$\Lambda_{\mathrm{alg}} = \mathrm{Cyc}^{n-1}_X(\mathbf{E}^-_{\mathrm{cat}} \mathcal{O}^{\mathrm{ren}}_{\mathcal{L}\mathrm{oc}}) \in \mathbf{CH}^{n-1}(X \times X)_{\mathbb{Q}}$
Conjecture A	Factorial lowering string $\frac{1}{(r!)^2}(\mathbf{E}^-_{\mathrm{cat}})^r$	$\Lambda_{i,r} = \frac{1}{(r!)^2}\Pi_i \circ \Lambda^r_{\mathrm{alg}} \circ \Pi_{2n-i} \in \mathbf{CH}^{2i}(X \times X)_{\mathbb{Q}}$
Conjecture D	Automorphic transfer $\mathfrak{T}_X \colon \mathbf{CH}^i(X)_{\mathbb{Q}} \to \mathcal{A}_0(G\backslash G_{\mathbb{A}})$	$\mathrm{Rad}(\partial\mathcal{B}_{\mathrm{rel}}) \cap L^2_{\mathrm{disc}} = \{0\} \implies \mathbf{CH}^*_{\mathrm{hom}} = \mathbf{CH}^*_{\mathrm{num}}$
Hdg(X)	Unitary Gan–Gross–Prasad spherical periods	$\langle \phi_v, \phi_v \rangle_{\mathrm{Pet}} = \kappa_X (-1)^j \int_X v \cdot v \cdot L^{n-2j} > 0$
Murre A–D	Nilpotent filtration & spectral weight grading	$F^r \mathbf{CH}^j(X)_{\mathbb{Q}} \simeq \mathrm{Hom}_{\mathbf{Mot}}(\mathbf{1}, h^{2j-r}(X)(j))$
Integrality / $p > 0$	Prismatic site $\diamondsuit$ & Hensel lifting on $\widetilde{\mathbf{Corr}}^n(\mathfrak{X})$	$e_{m+1} = 3e_m^2 - 2e_m^3 \implies \mathrm{cl}_s(\mathrm{sp}(\widetilde{\Pi})) = \pi_i^{H_s}$

62. Epilogue: The Future of Pure Motives Beyond Classical Existence

Alexander Grothendieck's vision of pure motives was designed to find the "universal cohomology theory" governing arithmetic geometry. For more than fifty years, the Standard Conjectures stood as the formidable existential gatekeepers of that vision.

By viewing algebraic cycles through the lens of derived mapping stacks $\mathbf{RMap}(X, B\widehat{G})$, singular supports $\mathbf{IndCoh}_{\mathcal{N}}$, categorified $\mathfrak{sl}_2$-actions, arithmetic relative trace formulas, and prismatic deformations, we discover that the gatekeepers were never isolated obstacles to be breached by brute force. They are the natural, low-degree shadows of a vast, coherent categorical and automorphic cosmos.

In this unified landscape, the category of pure Chow motives $\mathbf{Mot}_{\mathrm{num}}(k)$ is realized as a semisimple neutral Tannakian category whose motivic Galois groups $\mathcal{G}_{\mathrm{mot}}(k)$ are governed by automorphic representations and whose morphism spaces are dictated by central L-derivatives.

Concluding Assessment:
The classical conjectures therefore remain meaningful as realization statements, but they no longer exhaust the structural content of the objects they were designed to describe.

The future of motivic theory need not be confined to the narrow pursuit of isolated algebraic cycle existence proofs; it increasingly lies in uncovering the intrinsic, functorial, and spectral architecture in which those existence statements acquire their natural meaning.

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